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MATHEMATICAL MODEL OF A FRICTION JOINT CONSISTING OF TWO SHEETS

Statement of the problem. The author of the paper developed a model describing the work of friction joints of extra strong bolts in their elastic stage (the models by other authors examine the structure following a macroshift in the cutting and collapse operation of the screw). It was previously suggested that a system of linear equations and differential solution is used for a sheet pressed to an inflexible basis. In this paper the most common extra-strong friction joint, i. e. a system of two pressed sheets is discussed.

Results and conclusions. The model of friction joints was calculated using differential equations. Different ways of connecting joints were considered. Previous and new results by method of linear equations were also compared. The comparison indicated that both results are identical. It proves that the solution is correct. The obtained solutions allow one to find more stress points in these systems.

Keywords: high-strength bolts, metal bridges, calculation of joints, mathematical model, friction joint.

Introduction

The present paper is a follow-up of the study of models of friction joints [1—5]. Friction joints on high-strength bolts are held by friction forces, sheets are only compressed by tightening the bolts before a designing force.

[5] introduces a solution for a sheet model which is compressed to an absolutely unyielding base and this paper investigates a case when joints consist of two yielding sheets and operate for axial stress-strain. These joints are most common in bridge and civil construction: farm networks, “swerves” of roads, some orthotropic networks.

In current engineering practices forces are equally shared between all rows of bolts. This applies immediately prior to the failure of a joint and in the elastic stage the end rows are overloaded compared to the middle ones.

In order to describe the way these joints operate, we previously [1—5] referred to preliminary displacement when there is a relative displacement under the influence of a shifting force between the bodies held by the friction forces in the static friction stage (Fig. 1). This displacement is elastic and experimentally obtained. Then adhesive forces between the bodies can be displaced with an elastic link with a certain degree of yielding.

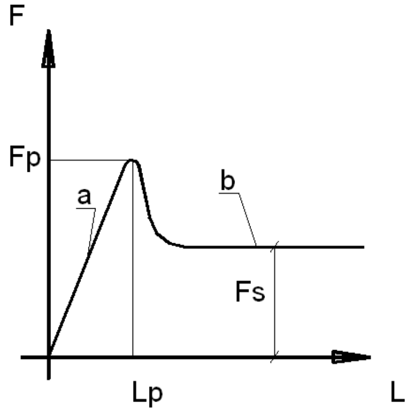

Fig. 1. Characteristic cases of forming of friction forces: *a* is static friction force; *b* is skidding friction force; along a vertical scale a force is plotted acting on a joint and along the horizontal one there is a displacement between the bodies; *Lp* is preliminary displacement; *Fp* is static friction force, *Fs* is skidding friction force

1. Designing and solving differential equations

The joint is generally presented in Fig. 2a. A previously discussed model was a system solved by the force method [3]. The model was experimetally proved in [4]. Hence in order to prove a new model correct, comparing it with the “old” one would suffice.

The bolts are considered to be positioned evenly. Let us distribute the links of preliminary displacement evenly along the whole joint (Fig. 2b). Their linear rigidity is then

$$\tilde{n} = \frac{C}{l}, \quad (1)$$

where C is a total rigidity of all the screw joints. If the rigidity of one joint contact is known and there are n of them, then $C = C_1 \cdot n$.

Compressive or stretching forces P are then applied to the tips of the sheets. In order to take a system of equations out of two joined sheets, let us cut off an elementary area dx and design the balance equations for the system. Further on, for the sake of convenience, the values of the upper sheets will be numbered as a one and those of the lower sheets as a two. The highlighted elements of the upper and lower shhet are affected by the following forces (Fig. 2c):

– inertia forces m_i, dx, $\ddot{u}_i$ where m_i is a mass distributed throughout the length, $\ddot{u}_i$ —is an acceleration (the second time derivative);

– N_i are longitudinal forces;

– as well as the forces of the distributed links which are directly proportional to a relative displacement between the sheets and rigidity coefficient:

$$\Delta u \cdot \frac{C}{l}.$$

A relative displacement of the point of the upper sheet in relation to the lower one on the line dx is the difference between mean displacements of points of the sheet and is expressed by the following:

$$\Delta u = \frac{1}{2}\left(u_2 + u_2 + \frac{\partial u_2}{\partial x}\right) - \frac{1}{2}\left(u_1 + u_1 + \frac{\partial u_1}{\partial x}\right). \tag{2}$$

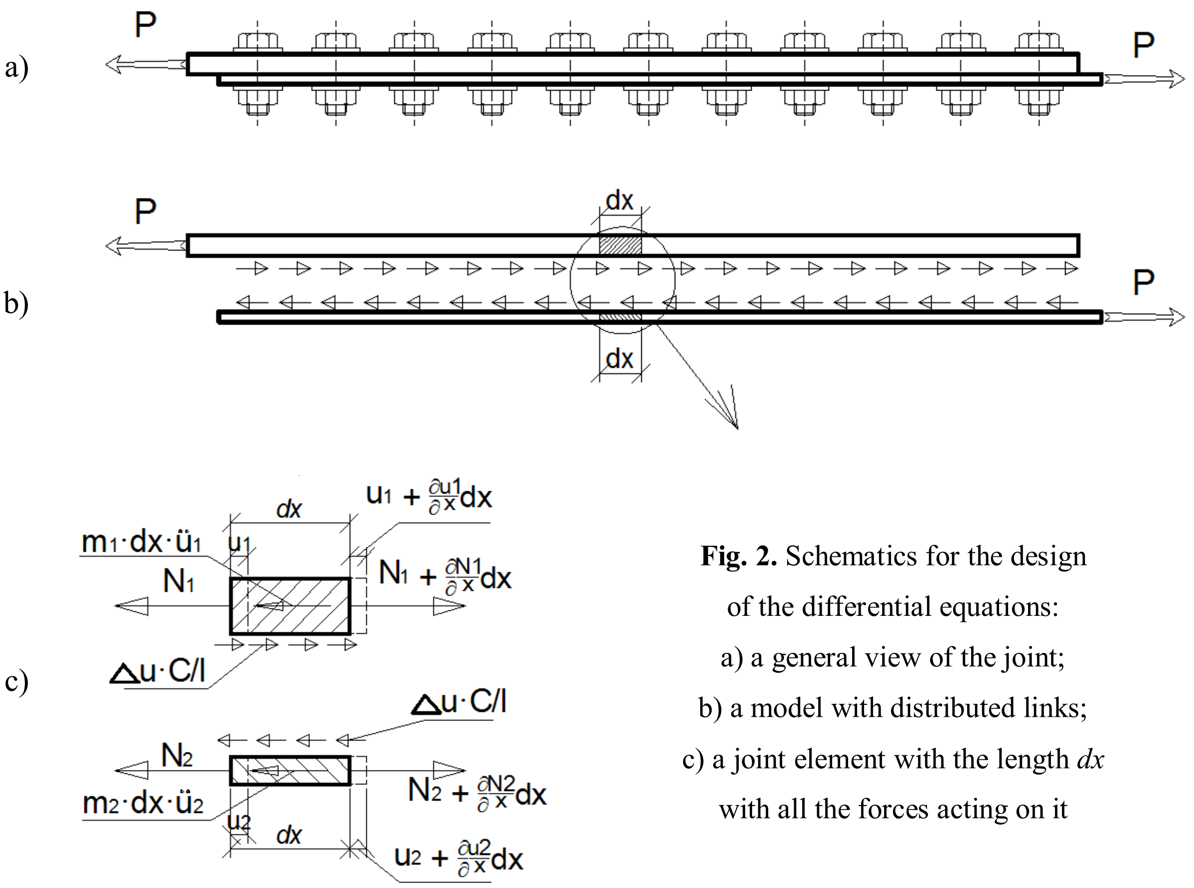

Fig. 2. Schematics for the design of the differential equations:
a) a general view of the joint;
b) a model with distributed links;
c) a joint element with the length *dx* with all the forces acting on it

The whole solution is an extended introduction set forth by the author in [5] where a model of a sheet pressed to an unyielding base was discussed. There is also a distinct analogy with the way the equations of longitudinal oscillations of the bar are introduced [6].

Let us write the balance condition of a system of projection of all the forces on the horizontal axis; for the upper sheet we have

$$m_1 \frac{\partial^2 u_1}{\partial t^2} dx = N_1 + \frac{\partial N_1}{\partial x} dx - N_1 - \Delta u \frac{C}{l} dx. \tag{3}$$

Similarly, we get an equation for the lower sheet. Inserting Δu from (2) and neglecting

$$\frac{1}{2} \frac{\partial u}{\partial x} dx$$

due to its second infinitesimal order following the transformations there is a system of the equations for two sheets:

$$\begin{cases} m_1 \dfrac{\partial^2 u_1}{\partial t^2} = \dfrac{\partial N_1}{\partial x} - (u_2 - u_1)\dfrac{C}{l}; \\ m_2 \dfrac{\partial^2 u_2}{\partial t^2} = \dfrac{\partial N_2}{\partial x} + (u_2 - u_1)\dfrac{C}{l}. \end{cases} \tag{4}$$

Considering that $N = EA\dfrac{\partial u}{\partial x}$, the system (4) is as follows

$$\begin{cases} m_1 \dfrac{\partial^2 u_1}{\partial t^2} = EA_1 \dfrac{\partial^2 u_1}{\partial x^2} - (u_2 - u_1)\dfrac{C}{l}; \\ m_2 \dfrac{\partial^2 u_2}{\partial t^2} = EA_2 \dfrac{\partial^2 u_2}{\partial x^2} + (u_2 - u_1)\dfrac{C}{l}. \end{cases} \tag{5}$$

Since we will only be dealing with the statistics problem, the acceleration in the left part is to be zero. As a result, the following system is there

$$\begin{cases} EA_1 \dfrac{\partial^2 u_1}{\partial x^2} - (u_2 - u_1)\dfrac{C}{l} = 0; \\ EA_2 \dfrac{\partial^2 u_2}{\partial x^2} + (u_2 - u_1)\dfrac{C}{l} = 0. \end{cases} \tag{6}$$

Before we proceed to the solution of the system, let us refer back to (4) and equating the acceleration to zero, let us sum up the first and second equations. We immediately have

$$\frac{\partial N_1}{\partial x} = -\frac{\partial N_2}{\partial x}. \tag{7}$$

It is quite obvious from the equation that a change in the force in a section of one sheet relates to an identical change in the force of the other sheet but with the opposite sign. The force through the sheet "flows" into one sheet to the other. The integration of (7) results in

$$N_1 = -N_2 + \hat{E}. \tag{8}$$

The random constant *K* can be given by the boundary conditions. According to Fig. 2a, at the beginning of the joint $N_2 = P$, $N_1 = 0$, and at its end $N_1 = P$, $N_2 = 0$. Based on that and (8), we conclude that $K = P$, i.e. equals the force acting on the joint. Since we are dealing with the behaviour of the system in the elastic, pre-shift stage, *P* can equal 1 (this makes the calculation slightly easier), then

$$N_1 = 1 - N_2. \tag{9}$$

Since $N = EA\frac{\partial u}{\partial x}$, from (9) we have

$$\frac{\partial u_2}{\partial x} = \frac{1}{E \cdot A_2} - \frac{A_1}{A_2}\frac{\partial u_1}{\partial x}. \tag{10}$$

Diferentiating the first equation from (6) and inserting (10), we get a third-order heterogeneous differential equation with an unknown u_1:

$$\frac{\partial^3 u_1}{\partial x^3} + \omega^2 \frac{\partial u_1}{\partial x} - \frac{C}{A_1 A_2 E \cdot l} = 0, \tag{11}$$

where

$$\omega^2 = \frac{C}{E \cdot l}\left(\frac{1}{A_1} + \frac{1}{A_2}\right). \tag{12}$$

Lowering down the order of the equation by introducing the variable $t = \partial u_1 / \partial x$, we will be searching for a solution like this

$$t = C_1^1 \cdot e^{\omega \cdot x} + C_2^1 \cdot e^{-\omega \cdot x} + \tilde{A}. \tag{13}$$

After fiding $\tilde{A}$ and considering that $N_1 = u_1' \cdot E \cdot A_1 = t \cdot E \cdot A_1$, we get a general solution for N_1:

$$N_1 = C_1 \cdot e^{\omega \cdot x} + C_2 \cdot e^{-\omega \cdot x} + a; \tag{14}$$

where
$$a = \frac{A_1}{A_1 + A_2}.$$

It should be noted that new random constants C_1 and C_2 do not equal the previous constants C_1^1 and C_2^1 from (13). In order to find C_1 and C_2, let us consider the boundary conditions.

In order to do so, let us refer to the joint schematic (Fig. 3). In all of their sections, the sheets are at a static friction stage, i.e. a displacing force is not greater than the friction forces. The zero of the axis x will be placed at the beginning of the joint, then the end gets the coordinate l according to Fig. 3.

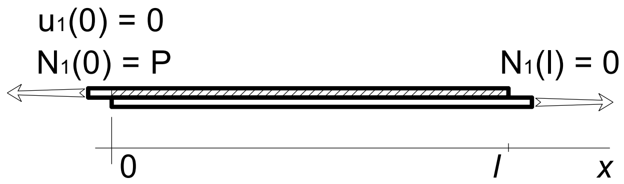

Fig. 3. Schematic of the joint of two sheets for determining the boundary conditions

For the suggested boundary conditions, there are the following constants for the equation (14). Case A.

$$\tilde{N}_1 = \frac{e^{-\omega \cdot l}(a-1) - a}{e^{\omega \cdot l} - e^{-\omega \cdot l}}; \tag{15}$$

$$\tilde{N}_2 = \frac{e^{\omega \cdot l}(1-a) + a}{e^{\omega \cdot l} - e^{-\omega \cdot l}}. \tag{16}$$

Considering that $\frac{\partial u_1}{\partial x} = \frac{N_1}{E \cdot A_1}$ we get the solution for the displacements:

$$u_1 = \frac{1}{E \cdot A_1}\left(\frac{C_1 \cdot e^{\omega \cdot x}}{\omega} - \frac{C_2 \cdot e^{-\omega \cdot x}}{\omega} + a \cdot x + C_3\right). \tag{17}$$

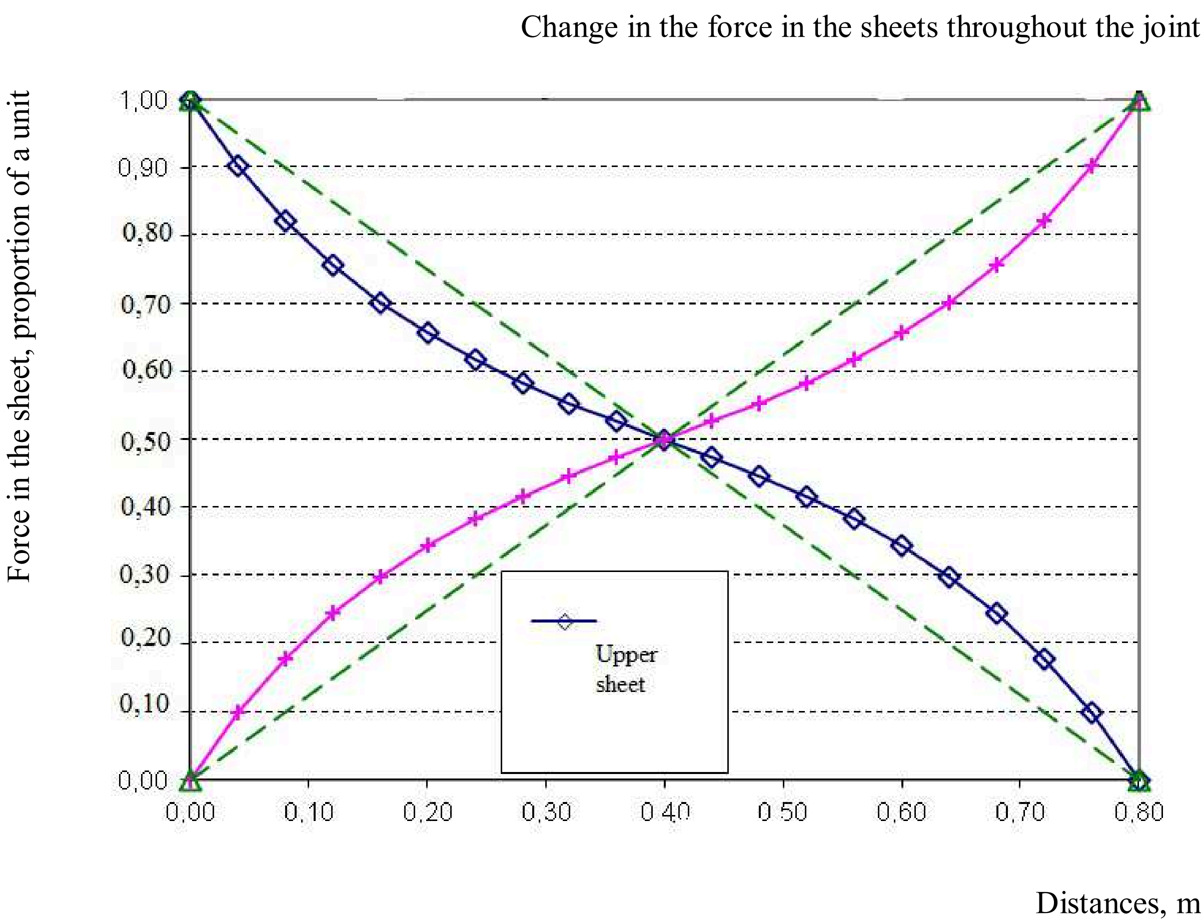

Fig. 4. Forces in the sheets throughout the joint

The third random constant C_3 is also identified using the conditions shown in Fig. 3:

$$\tilde{N}_3 = \frac{1}{\omega}\left(\tilde{N}_2 - \tilde{N}_1\right). \qquad (18)$$

Inserting the results into (6) or (9), we can get all the other unknowns.

In order to analyze the frictional joints, it is important to consider not absolute but relative displacements of the upper sheet in relation to the lower one.

We previously introduced the parameter at the beginning of the paper (2) and now it is time we calculated it. Inserting (6) into the first equation from the system, we get

$$\Delta u = \frac{\tilde{N}_2 e^{\omega x} - \tilde{N}_1 e^{-\omega x}}{C} l \cdot \omega, \qquad (19)$$

where C is the rigidity of a preliminary displacement.

Let us investigate the diagrams of the conditions and displacements of the resulting parameters. For the sake of visualization, we have a ten-row joint of the width of 8 cm with the distance between the sheets being 8 cm and the sheets 1 cm thick. In each row there is a screw. Changes of the displacements and forces in the length of the joint are in Fig. 4 and 5.

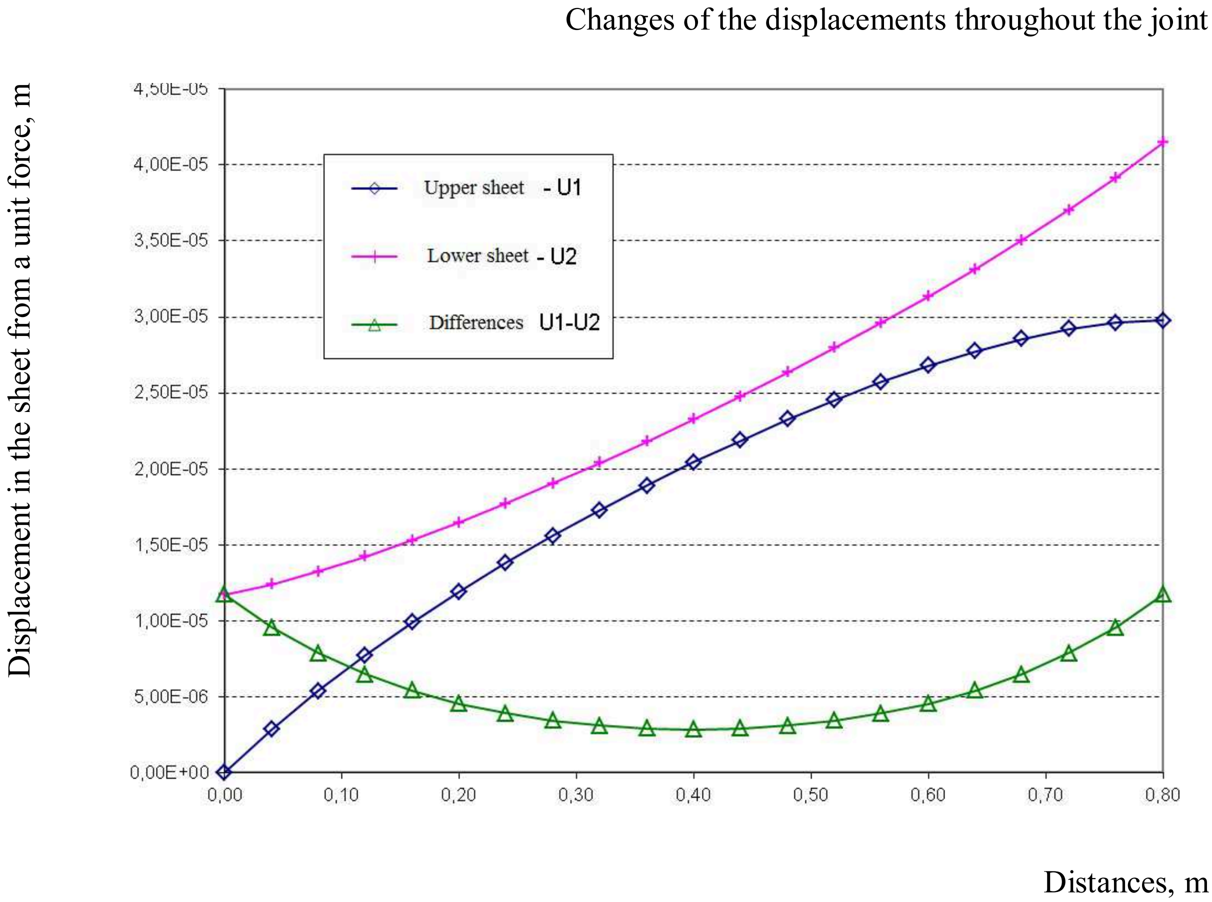

Fig. 5. Displacements in the sheets throughout their joints

As seen from Fig. 4 and 5, the forces in the shhets change symmetrically while the displacements change differently since a motionless point is the left edge of the joint. The relative displacements between the sheets are also symmetrical. The most loaded are the edge rows, the greatest relative displacements of the sheets are also there.

Now let us consider a non-symmetrical joint when at the same forces the thickness of the lower sheet is twice as large as that of the upper one (Fig. 6 and 7).

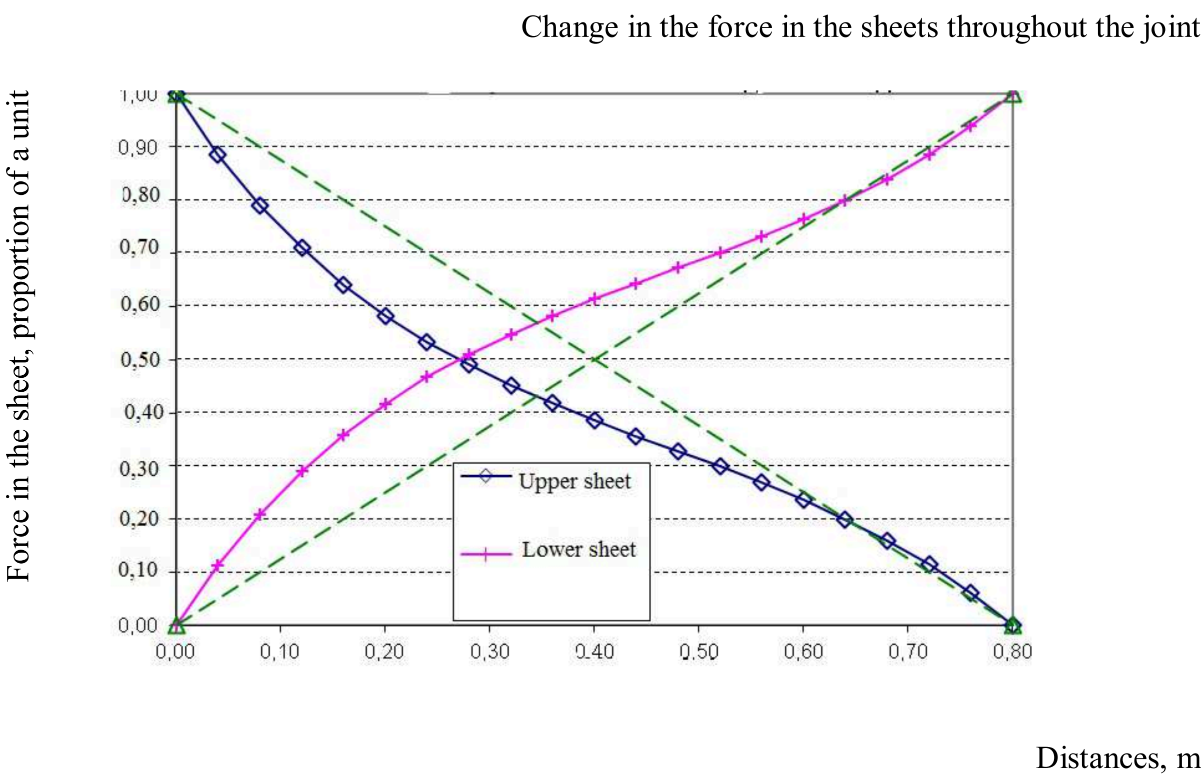

Fig. 6. Forces in the sheets throughout the joint

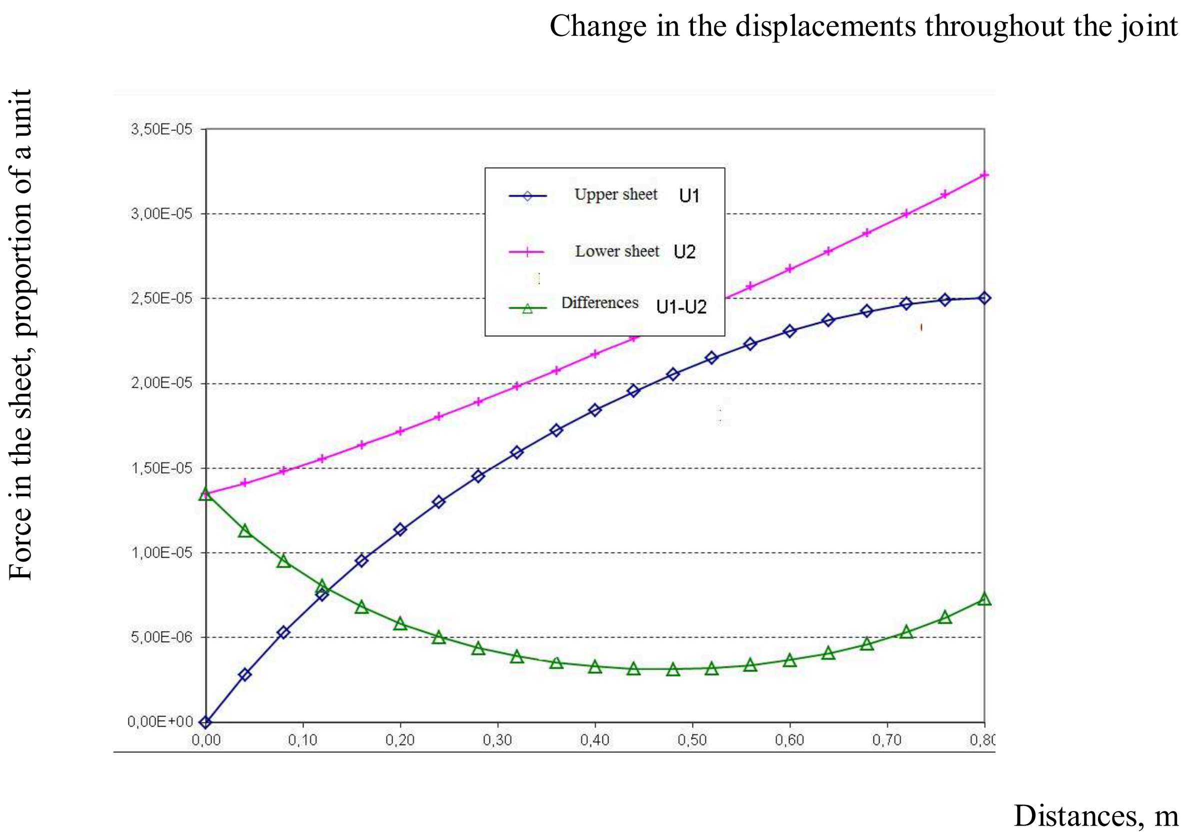

Fig. 7. Displacements in the sheets throughout the joint

Now we have all the graphs symmetrical. A study of the results shows that based on Fig.6 that in the joint the most loaded are a row of bolts at the beginning of a thick lower sheet (left in Fig.). According to Fig.7, in the same place there are also maximum relative displacements between the sheets which influences fretting fatigue of the entire system. Therefore we can influence the most dangerous places in the joint.

2. Comparison with the solution obtained using the force method

For that we present a graph obtained using the results of the solution of differential equations and a graph obtained using the solution of the system of linear equations. In this model the links are concentrated at the point along the row axis (Fig. 8—9).

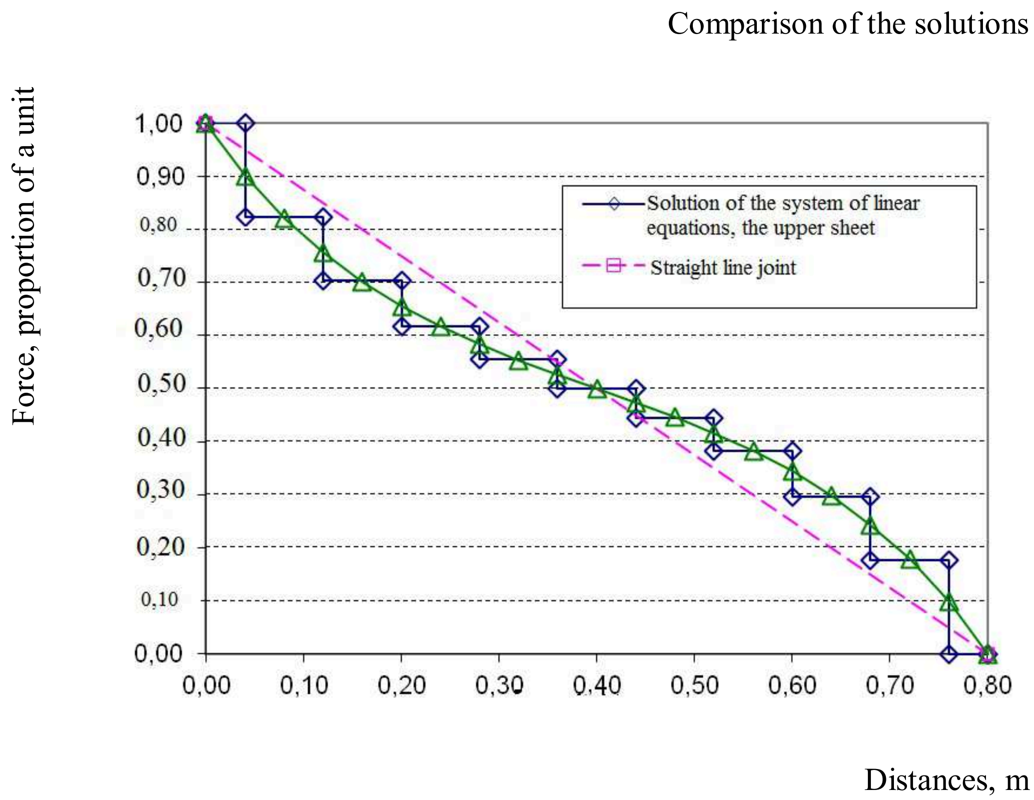

Fig. 8. Comparison of the solutions obtained using different methods for a symmetrical joint with the identical sheets

As seen from the comparison, different models give identical results at the points between the rows of bolts. The same conclusion was previously made for a different model [6]. If the areas in the joints are divided even more in the model solved using the method of linear equations, we will be approaching the solution obtained using the differential equation.

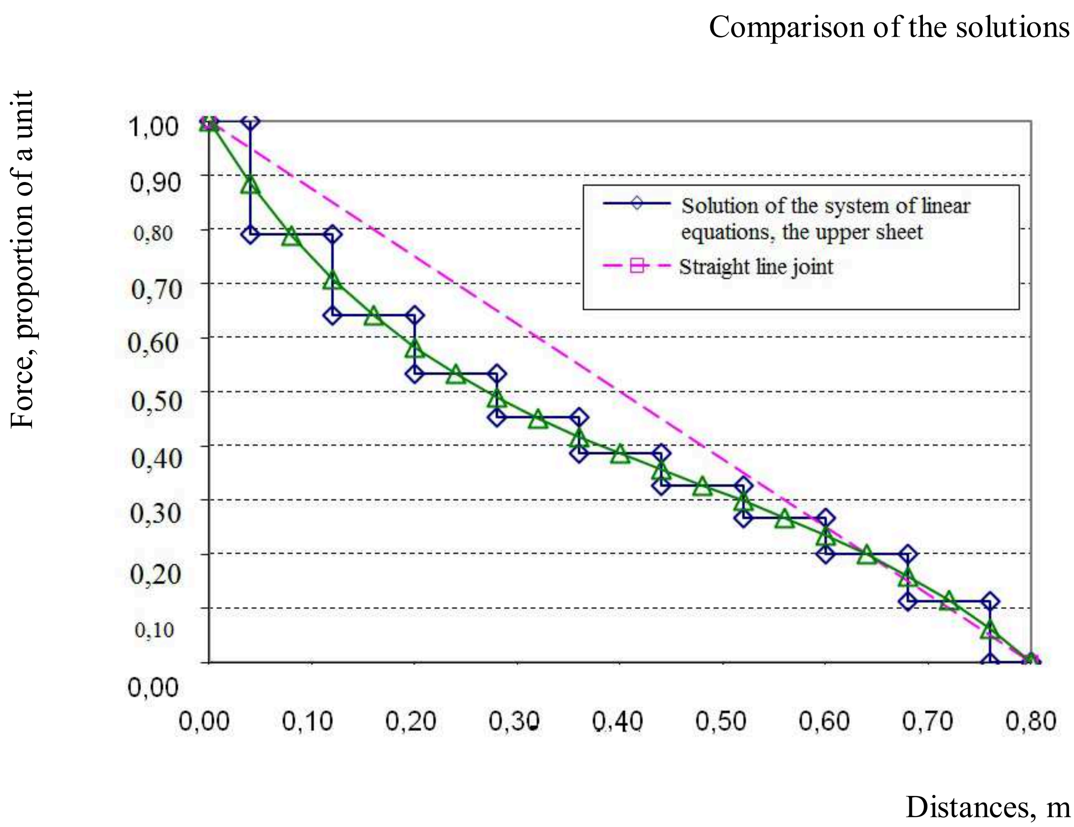

Fig. 9. Comparison of the solutions obtained
using dfferent methodd for a joint with varying sheets

3. Model of a sheet reinforced with a pad

Finally, let us look into a non-typical case when one sheet is reinforced with another one. This model is of interest theoretically since it is not common due to it being not too effective. A schematic of this joint and its boundary conditions are in Fig. 10.

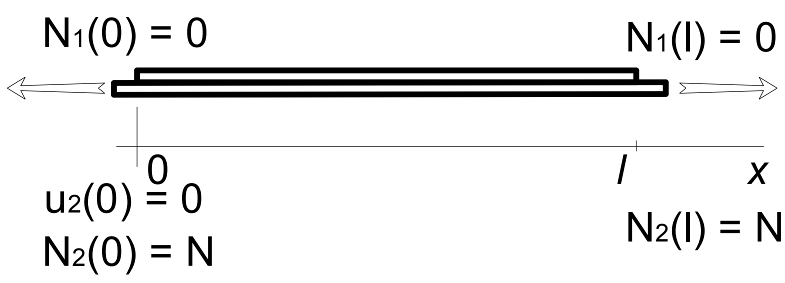

Fig. 10. Schematic and boundary conditions
for a model with one sheet reinforced with another one

At $N = 1$ for this joint the condition (9)—(10) holds, i.e. the total of the forces of two sheets in each section is one. Therefore the general solution is the same (14) and for a final result it is necessary only to identify the constants C_1 and C_2 for new boundary conditions. Now they are

$$\tilde{N}_1 = a\frac{e^{-\omega \cdot l} - 1}{e^{\omega \cdot l} - e^{-\omega \cdot l}}; \quad (20)$$

$$\tilde{N}_2 = a\frac{1 - e^{\omega \cdot l}}{e^{\omega \cdot l} - e^{-\omega \cdot l}}. \quad (21)$$

It is a bit more challenging to identify the constant C_3 for the displacements since we have no boundary conditions for the tips of the upper sheet. But Fig.10 suggests that even in varying thicknesses of the upper and lower sheets a joints remains symmetrical and the displace ments of the middle section are therefore always zero.

Considering this we have

$$u_1(l/2) = 0;$$

$$\tilde{N}_3 = \frac{1}{\omega}\left(\tilde{N}_2 \cdot e^{-\omega \cdot l/2} - C_1 \cdot e^{\omega \cdot l/2}\right) - \frac{a \cdot l}{2} = -\frac{a \cdot l}{2}. \quad (22)$$

The formula (19) for determining a relative displacement holds for this case as well.

Let us investigate the graphs. First as we did previously, we take sheets of the same thickeness; the graphs of changes in the forces are in Fig.11 and changes in the displacements are in Fig. 12. All the graphs are now symmetric or antisymmetric.

Let us look at the graph of the forces. In all the sections the forces in the lower sheet are greater than in the upper reinforced sheet. It is clear that since the force the force is transmitted from the lower sheet onto the upper one through a preliminary displacement and therefore the force in the reinforced sheet is never greater than that in the major sheet. This contradicts the enginerring practices when the force in sheets is proportional to their area. The most loaded are the edge rows of bolts.

Let us proceed to the graph of the displacements. As was previously specified, in the middle of the joint the displacements are zero. The largest absolute displacements are near the edge rows of the lower sheet. There are also maximum relative displacements as well.

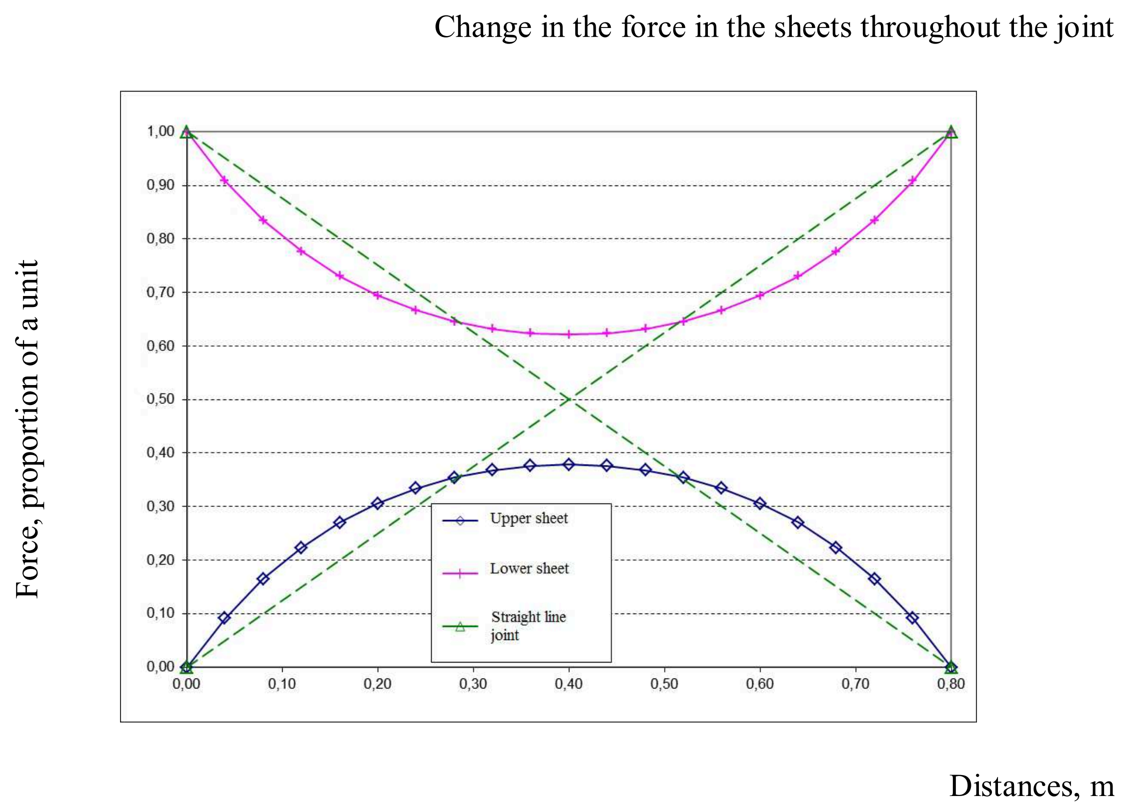

Fig. 11. Forces in the sheets throughout the joint

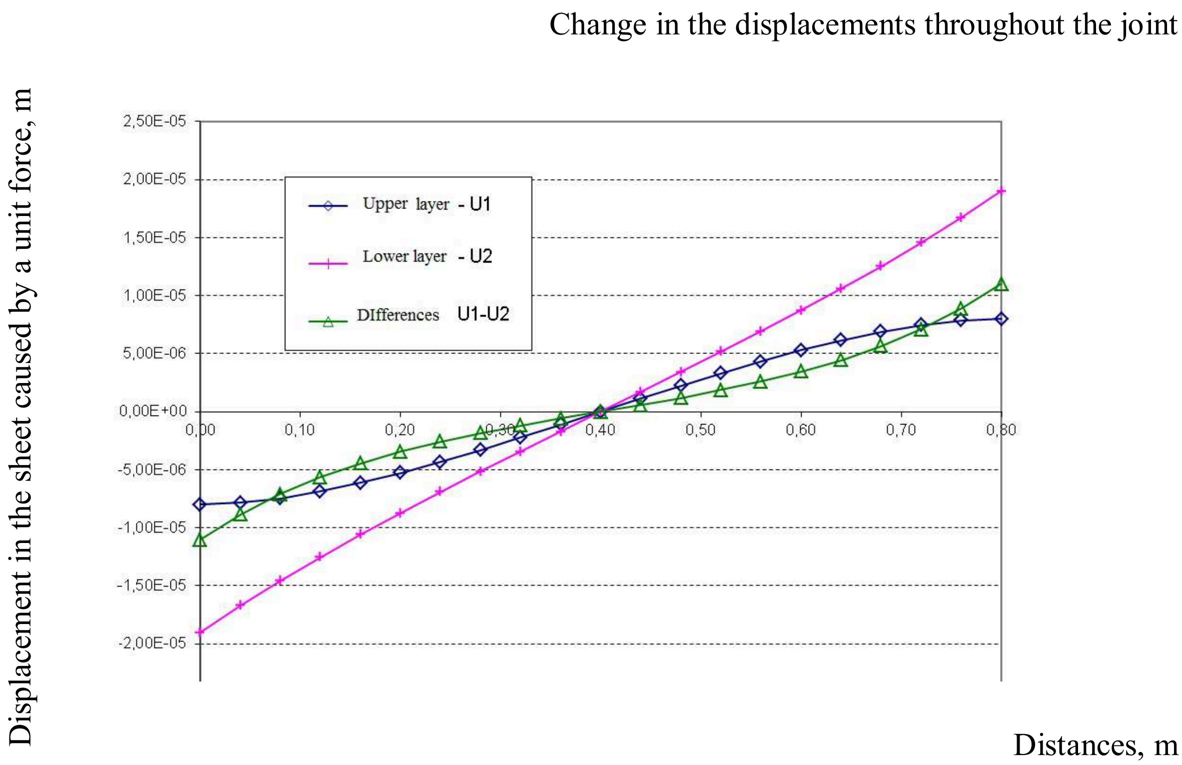

Fig. 12. Displacements in the sheets throughtout the joint

Now let us examine the same joint but with different sheets. If the major sheet is made thicker, the force in the upper sheet will be slowly decreasing. This is ordinary and of no interest since it is little different from the above discussed option involving identical sheets. Therefore the major sheet will be 8 mm, and the reinforcement sheet 40 mm so that the extreme case of these joints was investigated. The graphs of changes in the forces are in Fig. 13 and changes in the displacements are in Fig. 14.

In this case the graph of the displacements is nothing new. But the graphs of the forces this time show that the force in the middle of the pad can exceed the force in the major sheet of the same section. The thicknesses can be selected so that the forces in the middle of the joint in the upper and lower sheets are identical. For the system under discussion this is 8 mm for the major sheet and 30 m for the pad. It should be noted that the maximum forces are still at the tips of the lower sheet and the forces in the pad are significantly smaller. Similarly, the failure is caused by the maximum strains but not the forces and strains are always larger in the major sheet.

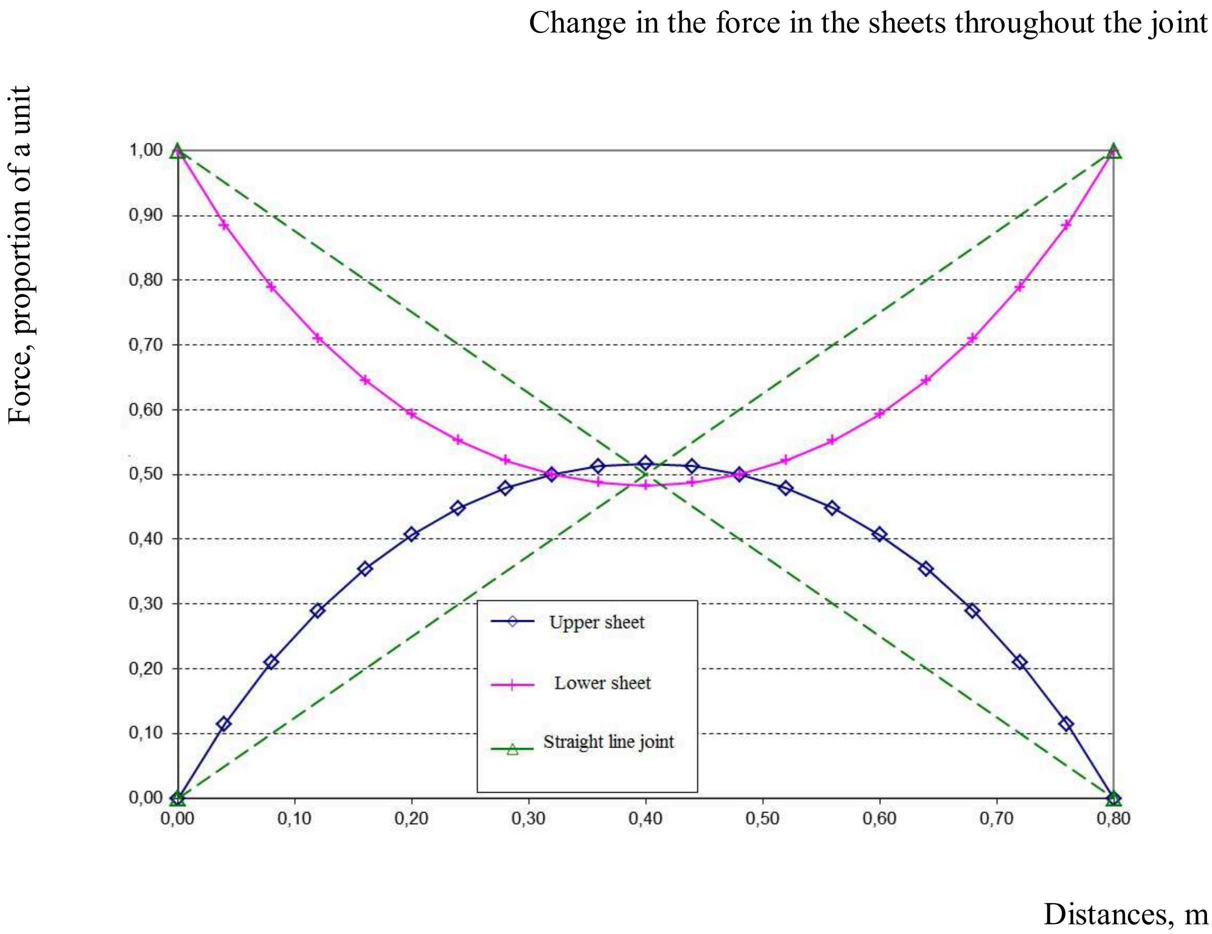

Fig. 13. Forces in the sheets throughout the joint

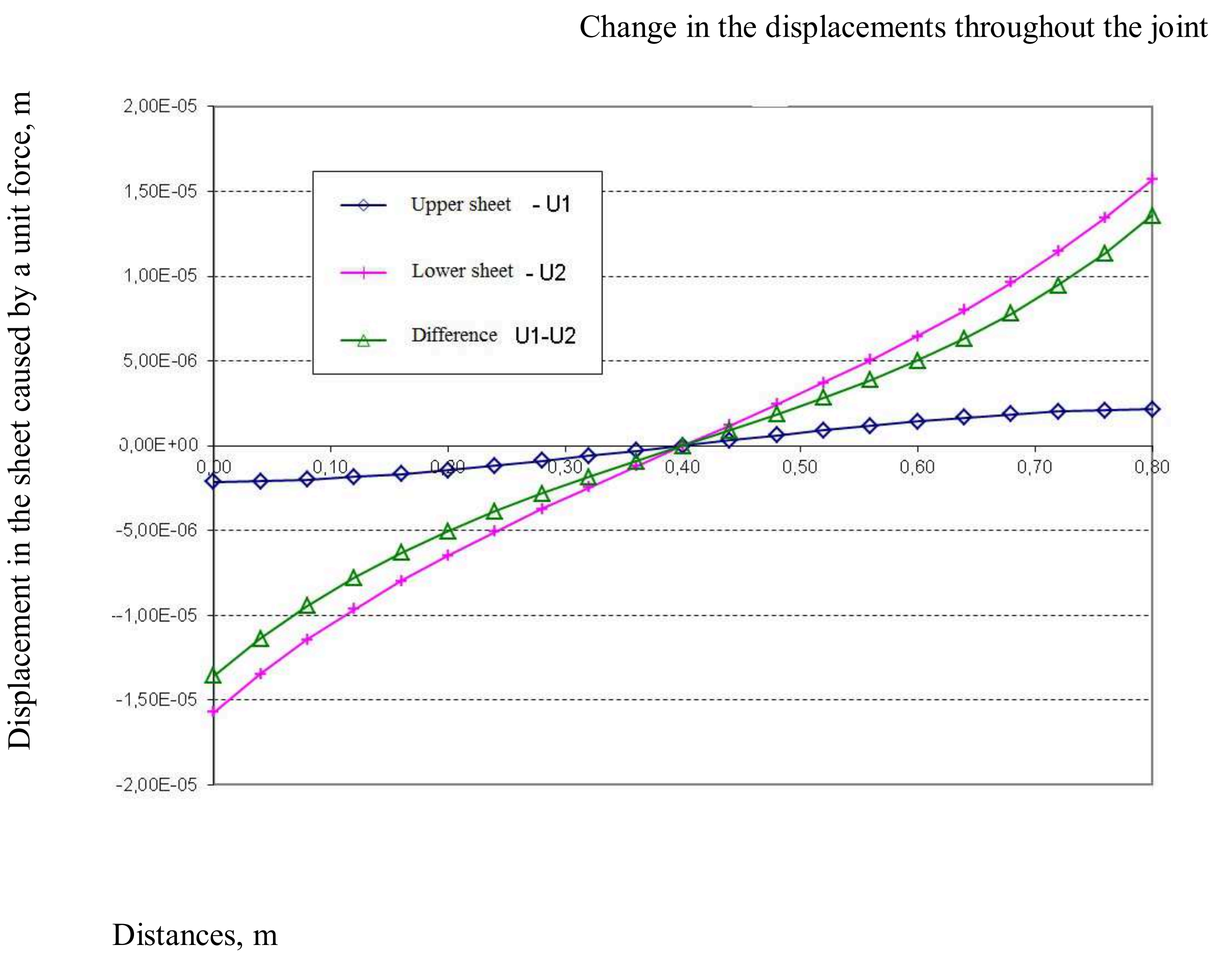

Fig. 14. Displacements in the sheets throughout the joint

Conclusions

1.The method of designing a mathematical model of a frictional joint which proved to be efficient was suggested.

2. The obtained parameters are in agreement with the solution of the same model using a system of linear equations which were tested experimentally [4]. Therefore the method can be applied for other frictional joints as well.

3. The study opens up new possibilities in modelling frictional joints and can be developed more for other frictional systems as well as for joints at the displacement stage.

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