BUILDING MECHANICS

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FLEXIBILITY MATRIX OF AN ELASTIC BASE IN ITS INTERACTION WITH A LOADED PLATE RESTING ON ITS SURFACE

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Statement of the problem. We consider the use of the finite element method in the problem of contact interaction of a thing loaded plate disposed on the Boussinesq's elastic half-space. The novelty of our setting is in the loading function which is supposed to be continuous and linear on every element of some contact region triangulation.
Results. We derive new formulas of calculation of the corresponding flexibility matrix for the Boussinesq's elastic half-space loaded by a piece-wise linear net function. The application of polar coordinate system in calculations connected with a net element makes these formulas new and more convenient in applications. Furthermore we use these formulas in the model of the interaction of the previous half-space and absolutely smooth punch loaded on its surface.
Conclusions. We suggest new effective formulas for the calculation of flexibility matrix in the case of piece-wise linear net loading function. The developed method is applicable to the calculation of plate deformations when it is disposed and loaded on the elastic half-space. This is demonstrated by investigation of 3D contact interaction between elastic half-space and absolutely smooth punch.

Keywords: flexibility matrix, plate, surface.

Introduction

According to the known approach [1—3] accepted in modelling interaction of a slab placed in the plane XOY of a rigid foundation under a load $p(x, y)$, the slab is considered in isolation and the foundation is replaced by its force reaction $q(x, y)$. In this case considering the hypothesis of Kirchhof-Lyav bending of a slab should agree with the following biharmonic equation:

$$D_1 \frac{\partial^4 W}{\partial x^4} + 2D_3 \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 W}{\partial y^4} = p - q, \quad (1)$$

supplemented by some boundary conditions. Here $W(x, y)$ is a vertical displacement of a middle surface of a slab and the slab itself can be equalled with this two-dimensional surface. Due to the complexity of a task at hand, particularly if a slab is a two-dimensional area of a non-standard shape, the solution is possible only using numerical methods. The main approaches in this case are either the finite difference method [2] or the finite element method [1, 3]. In both cases the most critical stage of the solution is determining the response of a foundation during its contact with a slab.

At this stage a certain model of an elastic foundation describing a connection between normal displacements $W(x, y)$ of a half-space surface and normal efforts acting on it is applied. This connection is specified as an integral operator

$$W(M) = \iint_D K(M, N) q(N) dN, \quad (2)$$

whose nuclear is critical to the model of elastic foundation. The Boussinesq nuclear is most common to use [1, 3]:

$$K(M, N) = B(M, N) = \frac{1-\nu^2}{\pi E |MN|} = \frac{1-\nu^2}{\pi E \sqrt{(x-u)^2 + (y-v)^2}},$$

which specifies a model of a homogeneous elastic half-space. We are going to look at it further on in the paper.

1. Problem of reducing a reaction of an elastic foundation to the nodes of a finite element discrete model of a slab. In a numerical solution of the above task of an elastic foundation instead of the function $W(x, y)$ of two variables its discrete approximation in the nodes of finite difference and finite element grid. In both cases on a slab surface of a foundation reaction should also be represented by a system of forces applied in the nodes of a corresponding grid. It is approximately achieved by considering another so-called dual grid [4—6] on the same two-dimensional area [4—6]. Each of its cells has a node of the original grid. Approximately applying the foundation reaction which is constant within each cell of a dual grid and choosing the grid itself so that the nodes of the original grid were in the gravity centres of the cells, the distribution of a reaction of a foundation along the nodes of the original grid can be fairly accurate which implements a discrete model of a slab. After that a slab is considered in isolation from a foundation. Besides the boundary task for the equation

(1) it is necessary to consider ordinary equations of balance of a solid body. It is used in [2] and some other papers.

However, this method is hard to use for nodes at the boundary of a slab and does not quite account for a quick growth of a foundation reaction at the boundary. In [1] for the finite element method on a triangular grid a piecewise-linear (linear within each triangular element) approximation of a foundation reaction. Due to that we need to reduce a variable load distributed along a triangular surface to three forces applied to the tops of a triangle. It is usually done by dividing the total load into three equal parts or reducing a load calculated using an adjacent third part of a triangle into its tops.

In this paper we suggest that a small triangular element of dividing a slab as an absolutely solid body and replace a load distributed along its surface with three forces applied at its tops so that they were equivalent to a load in the balance equations. I.e. the sum and both moments of these forces (the element is the plane XOY) should be the same as those of a distributed variable load.

When a variable is in the plane XOY, the load is specified by a linear function of the density $Z = aX + bY + C$ with values Z1, Z2, Z3 in the first, second and third tops of a triangle corresponding with the concentrated forces can be easily calculated using the formulas

$$P1 = (2Z1 + Z2 + Z3)S / 12,$$
$$P2 = (Z1 + 2Z2 + Z3)S / 12,$$
$$P3 = (Z1 + Z2 + 2Z3)S / 12,$$

where S is the area of a triangle.

Furthermore in order to determine the final force of a foundation reaction in the i-th node of a triangular node, the contribution of each triangle into the node having it as its top needs to be summed.

2. Flexibility matrix of an elastic foundation. Let there be some triangulation of a contact area of a slab and foundation with nodes at $\{M_i\}$, $i = 1, 2, \ldots, N$. Let us replace a variable along the surface of a foundation reaction $q(x, y)$ in Formula (2) by its piecewise-linear approximation $h(x, y)$ on the triangular grid of dividing a contact area. Remember that within each element of division of $h(x, y)$ is simply linear. Then within such functions it is natural to consider the basis $e_i(x, y)$ connected with the nodes M_i of a triangular grid. The elements of the basis are pyramidal functions:

1) linear on triangulation elements;
2) equalling 1 in the i-th node of the grid;
3) with a carrier equalling joining of all the triangles having the node i as their top.

Then

$$h(x, y) = \sum_{i=1}^{N} h(M_i) e_i(x, y).$$

This enables to consider $h(x, y)$ as an N-dimensional vector $h = \{h(M_i)\}$. Let $w=\{W(Mi)\}$ denote an N-dimensional vector of bendings of the slab surface in the triangulation nodes of a contact area. There is a matrix $B = (b\ (i, j))$ enabling the calculation of w using h. This matrix is called a flexibility matrix of a foundation. As a result we have an equality $w = Bh$ which yields a discrete equivalent of the Formula (2).

Each element of the matrix has some certain physical significance. I.e. $b(i, j)$ is vertical displacement of the node M_i under the effect of a pyramidal single load e_j applied in the node M_j.

Now for determining an element $b(i, j)$ of the matrix the displacement at the node i caused by a pyramidal basis load e_j applied at the node j needs to be calculated using the Formula (2). As e_j is linear on each triangular element of the grid and is not equal to zero only for those element of division that contain the node M_j as the top, for determining $b(i, j)$ double integrals just have to be calculated using a triangle of the product of the Boussinesq's nuclear and a linear function.

Some information about the formulas expressing $b(i, j)$ can be obtained using [1]. However no comments and complexity of the formulas themselves make them hard to use. Conversely, the below formulas appear to be easy to use in designing computer calculations.

3. Formulas for calculating elements of a flexibility matrix for the Boussinesq's nuclear. Let there be a triangle Δ with the tops at the points $A1$, $A2$, $A3$ (anti-clockwise) and a linear function $h(x, y) = Ax+By+C$ specifying a load on Δ and equaling 0 outside Δ. Let us calculate the displacement of the surface of an elastic semi-space at $S(x_0, y_0)$ caused by this distributed along Δ load $h(x, y) = Ax+By+C$. For that let us replace the start of the coordinate system at S. As a result $h(x, y)$ becomes $Ax+By+C_0$ where $C_0 = C+Ax_0+By_0$. After that we move on to a polar system of the coordinates with a polar at S.

Assuming $h(M)$ at a random $M(x, y)$ of the plane specified by the same formula

$$h(M) = h(x, y) = Ax + By + C_0,$$

we can write (Fig. 1) the following formula:

$$\int_{\Delta} \frac{h(N)}{|SN|} dN = -\int_{T_1} \frac{h(N)}{|SN|} dN + \int_{T_2} \frac{h(N)}{|SN|} dN + \int_{T_3} \frac{h(N)}{|SN|} dN.$$

I.e. the calculation of the integral Δ comes down to calculating the integral using three triangles $T1 = SA1A2$, $T2 = SA2A3$, $T3 = SA3A1$ with the top at S.

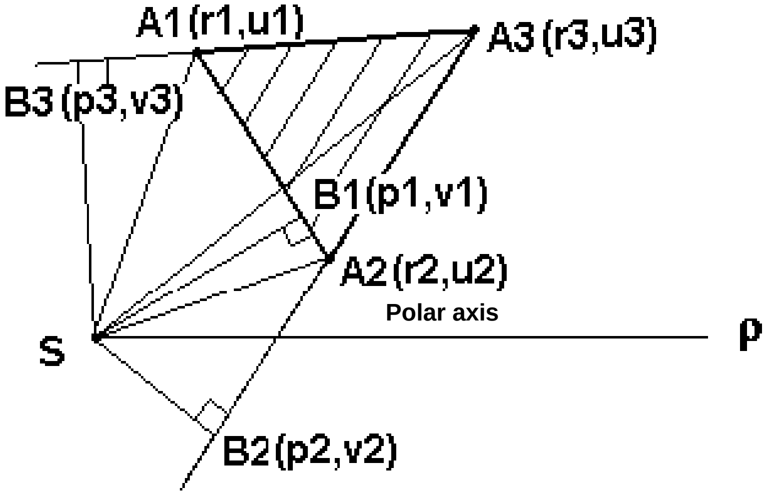

Fig. 1. Scheme of calculation of the integral using $\Delta A1A2A3$ in a polar coordinate system

Let ($p1$, $v1$), ($p2$, $v2$), ($p3$, $v3$) be polar coordinate of the perpendicular foundations projected from S onto the opposite side of the triangles $T1$, $T2$, $T3$ respectively. According to the definition ($p4$, $v4$) = ($p1$, $v1$). Let $u1$, $u2$, $u3$ be polar angles of the tops $A1$, $A2$, $A3$ respectively (according to the definition assume $u4 = u1$).

Then in the polar coordinate system using a triangle T_k, $k = 1, 2, 3$ with the top at (polar) S is

$$J_k = \int_{T_k} \frac{h(N)}{|SN|} dN = \int_{T_k} \frac{Ax + By + C_0}{\sqrt{x^2 + y^2}} dxdy = \int_{u_k}^{u_{k+1}} \Big(\int_0^{\frac{p_k}{\cos(\phi - v_k)}} \frac{Ar\cos\phi + Br\sin\phi + C_0}{r} rdr \Big) d\phi =$$

$$= A\int_{u_k}^{u_{k+1}} \cos\phi \int_0^{\frac{p_k}{\cos(\phi - v_k)}} rdrd\phi + B\int_{u_k}^{u_{k+1}} \sin\phi \int_0^{\frac{p_k}{\cos(\phi - v_k)}} rdrd\phi + C_0 \int_{u_k}^{u_{k+1}} d\phi \int_0^{\frac{p_k}{\cos(\phi - v_k)}} dr =$$

$$= \frac{A}{2} p_k^2 \int_{u_k}^{u_{k+1}} \frac{\cos\phi}{\cos^2(\phi - v_k)} d\phi + \frac{B}{2} p_k^2 \int_{u_k}^{u_{k+1}} \frac{\sin\phi}{\cos^2(\phi - v_k)} d\phi + C_0 p_k \int_{u_k}^{u_{k+1}} \frac{1}{\cos(\phi - v_k)} d\phi.$$

Using the formulas

$$\cos\phi = \cos(\phi - v_k)\cos v_k - \sin(\phi - v_k)\sin v_k,$$
$$\sin\phi = \sin(\phi - v_k)\cos v_k + \cos(\phi - v_k)\sin v_k,$$

we get

$$J_k = \frac{A}{2} p_k^2 \cos v_k \int_{u_k}^{u_{k+1}} \frac{d\phi}{\cos\ (\phi - v_k)} - \frac{A}{2} p_k^2 \sin v_k \int_{u_k}^{u_{k+1}} \frac{\sin(\phi - v_k) d\phi}{\cos^2(\phi - v_k)} +$$

$$+\frac{B}{2}p_k^2\cos v_k\int_{u_k}^{u_{k+1}}\frac{\sin(\phi-v_k)}{\cos^2(\phi-v_k)}d\phi+\frac{B}{2}p_k^2\sin v_k\int_{u_k}^{u_{k+1}}\frac{d\phi}{\cos\ (\phi-v_k)}+C_0p_k\int_{u_k}^{u_{k+1}}\frac{1}{\cos(\phi-v_k)}d\phi=$$

$$=C_k\int_{u_k}^{u_{k+1}}\frac{d\phi}{\cos\ (\phi-v_k)}+D_k\int_{u_k}^{u_{k+1}}\frac{\sin(\phi-v_k)}{\cos^2(\phi-v_k)}d\phi=\frac{C_k}{2}\ln\left|\frac{\sin(\phi-v_k)+1}{\sin(\phi-v_k)-1}\right|\Bigg|_{u_k}^{u_{k+1}}+$$

$$+D_k\frac{1}{\cos(\phi-v_k)}\Bigg|_{u_k}^{u_{k+1}}=\frac{C_k}{2}\ln\left|\frac{\sin(u_{k+1}-v_k)+1}{\sin(u_{k+1}-v_k)-1}\right|-\frac{C_k}{2}\ln\left|\frac{\sin(u_k-v_k)+1}{\sin(u_k-v_k)-1}\right|+D_k\frac{1}{\cos(u_{k+1}-v_k)}-$$

$$-D_k\frac{1}{\cos(u_k-v_k)},$$

where $$C_k=\frac{Ap_k^2}{2}\cos v_k+\frac{Bp_k^2}{2}\sin v_k+C_0p_k,$$

a $$D_k=\frac{Bp_k^2}{2}\cos v_k-\frac{Ap_k^2}{2}\sin v_k.$$

Therefore the displacement W at S using a linear load distributed along the triangle Δ is

$$W_S(\Delta)=\frac{1-v^2}{\pi E}(J_1+J_2+J_3).$$

Finally the element $b(i, j)$ of the flexibility matrix of the foundation equals the displacement in the node M_i of the basic load $ej(x, y)$. Hence

$$b(i,j)=\sum W_{M_i}\left(\Delta_k\right),$$

where all the division triangles of a contact area with the node M_j as their top are summed.

4. Results of a numerical experiment. Based on the formulas a calculation software was developed for elastic foundation slab which will be dealt with in a separate paper. Here we show the application of the formulas using the example of calculating a reaction of an elastic foundation as a rigid absolutely smooth punch is pressed into it vertically. For the solution we use the above discrete equivalent

$$Bh=w \qquad (3)$$

the equations (2) where $h = h(x, y)$ is piecewise-linear approximation of the density of a foundation reaction to be determined and w is the deformation (heaving) of the foundation. Small deformations are definitely investigated that in this example coincides with the heaving of the same rigid punch and are thus equal a constant under the punch and zero outside it. In order to identify h we only need to solve a linear system (3) with the known right part w.

Let us look at a square (according to the plan). We divided the square into equal triangles which were chosen to be completely symmetrical. In this case the matrix *B* becomes equally symmetrical to allow for solutions enabling a more simple Holetsky method compared to the Hausse method.

Fig. 2, 3 shows that in most of an elastic semi-space under the punch, the stress of the foundation, reaction of a foundation is small and almost constant. As we move from the city to the boundary, the foundation reaction starts dropping slowly and insignificantly, then decreases quite abruptly and then is slowly on the rise dramatically. Along the entire boundary of a square (except small parts adjacent to the top) the reaction is almost constant as well. As we approach the tops of the square, there is a sharp (about five time) growth of stress of the foundation reaction, which is consistent wit the known theoretical results. It should also be noted that on the diagonal lines adjacent to the inner nodes of the grid closest to the tops there are some smaller than in the centre but negative values of the reaction of the foundation. The latter suggests that the chosen model of the interaction of a rigid punch and elastic semi-space *without a semi-space detaching from the punch as w grounds outside its boundary* is not quite correct.

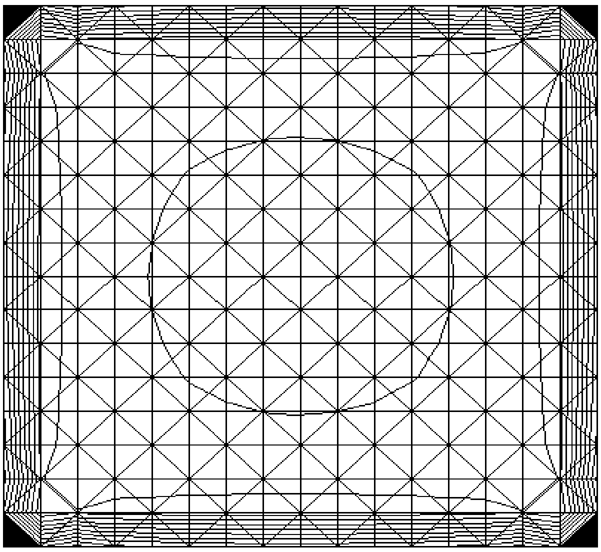

Fig. 2. Division structure and level lines of the reaction of the foundation

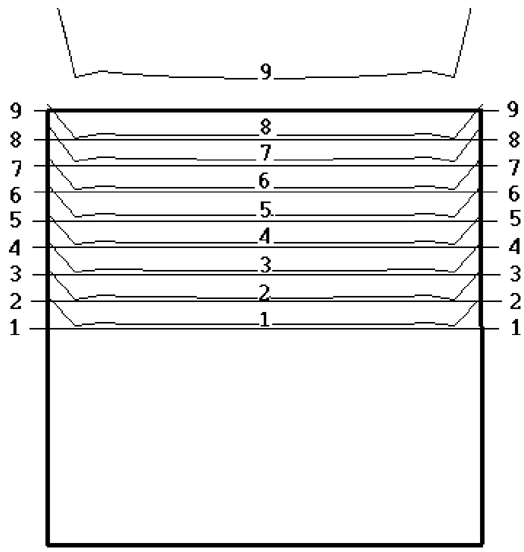

Fig. 3. Diagrams 1, 2, …, 9 of the reaction of the foundation under the foot of a rigid punch along the sections 1—1, 2—2, …, 9—9

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