BUILDING MECHANICS

UDC 624.04

M. N. Kirsanov[1]

A PRECISE SOLUTION OF THE TASK OF A BEND IN A LATTICE GIRDER WITH A RANDOM NUMBER OF PANELS

National Research University «Moscow Power Engineering University»
Russia, Moscow, tel.: (495)362-73-14, e-mail: c216@Ya.ru
[1]D. Sc. in Physics and Mathematics, Prof. of the Dept. of Robotics, Mechatronics, Dynamics and Strength of Machinery

Statement of the problem. A scheme of a flat elastic statically determined beam girder of a periodic structure is proposed. Both belts of the girder are rectangular, the lattice consists of stands and braces. Efforts in the rods, bend and a horizontal displacement of a moving support of the girder under an evenly distributed load are determined.
Results. Using the Moore integral, polynomial analytical dependencies are obtained for bends of the girder and critical efforts in certain rods on the number of panels, size and load. While generalizing particular solutions for a random number of panels, the inductive method is employed. It is noted that for the number of panels that are divisible by three the structure is kinematically changeable and the determiner of the equation system turns into zero. A field of possible velocities that correspond with that is presented.
Conclusions. For the suggested scheme of the lattice of the girder there are compact formulas that allow the evaluation of the rigidity, strength and durability of the elements of the structure. The condition for the kinematic changeability that was identified is a warning for a practicing engineering that there are some unacceptable options for the system parameters.

Keywords: lattice girder, bend, kinematic changeability, analytical solution.

Introduction. The number of statistically determined schemes of flat girders is limited [12, 13]. The number of girders where an analytical solutions is acceptable is even more so [1—3, 9, 10, 14, 15, 19, 20]. Analytical solutions that yield accurate results are known, but not all of them result in compact and easy-to-use formulas [7]. The advantages of accurate formula solutions for practical calculations, designing new structures and analyzing operational characteristics of existing ones are unquestionable. They can be used as test solutions for checking numerical methods or as simple evaluation formulas for a model of a structure.

© Kirsanov M. N., 2018

1. Scheme of a girder. A girder has a horizontal lower and an upper skew belt. The lattice consists of bars and braces that take up two or three panels (Fig. 1).

The girder with n panels in the half of the span contains $4n + 2$ hinges and $m = 8n + 4$ rods. In order to determine a bending and displacement of the support, it is necessary to determine the efforts in the rods of the girder. For symbolic transformations the method of of cutting out the nods is most suitable which mathematically involves a solution of a system of equilibrium equations for the nodes in the projections onto the coordinate axis. A matrix of the equation system consists of a direction cosine of efforts joined with the nodes. The direction cosines are determined using the coordinates of the end of the rods. The start of the coordinates can be conveniently placed in the left moving support (Fig. 2).

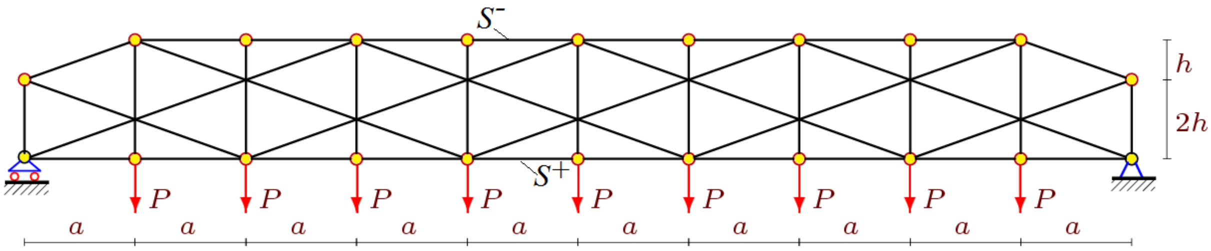

Fig. 1. Girder. General view at $n = 5$

The coordinates of the hinges of the girder are as follows:

$$x_i = (i-1)a, \quad y_i = 0, \quad i = 1,\ldots,2n+1,$$
$$x_{i+2n+2} = ia, \quad y_{i+2n+2} = 3h, \quad i = 1,\ldots,2n-1,$$
$$x_{2n+2} = 0, \quad y_{2n+2} = y_{4n+2} = 2h, \quad x_{4n+2} = 2na.$$

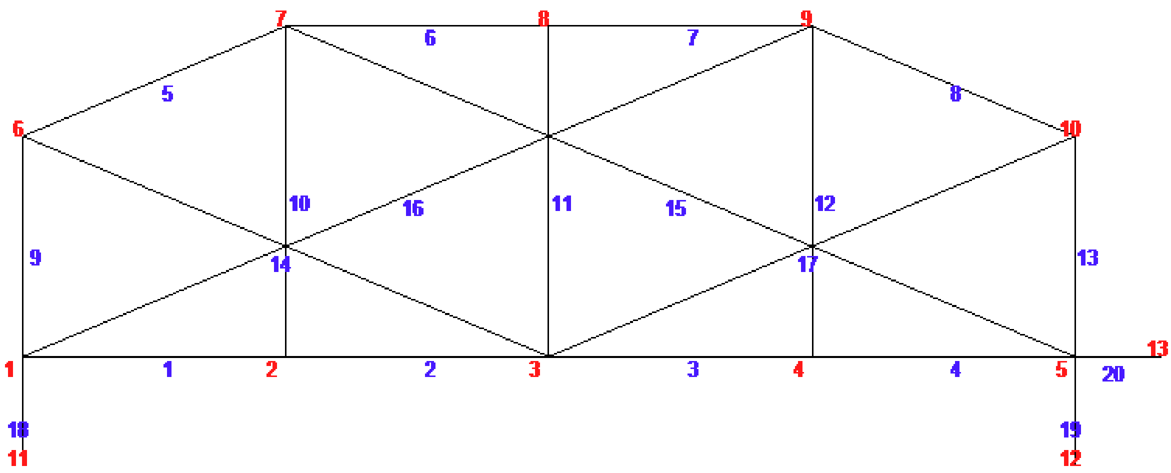

Fig. 2. Numbering of the nodes and rods in the girder at $n = 2$

The order of joining of the rods and hinges is specified using special vectors $\overline{q}_i$, $i=1,...,m$ containing the numbers of the ends of corresponding rods similarly to the way a list of the ribs of the graph in discrete mathematics [6]. The configuration of the belts and lattice is determined with the following vectors:

$$\overline{q}_i=[i,i+1],\ \overline{q}_{i+2n}=[i+2n+1,i+2n+2],\ i=1,...,2n,$$
$$\overline{q}_{i+4n}=[i,i+2n+1],\ i=1,...,2n+1,$$
$$\overline{q}_{6n+2}=[3,2n+2],\quad \overline{q}_{8n+1}=[2+4n,2n-1],$$
$$\overline{q}_{i+6n+2}=[2i+3,2i+2n+1],\ \overline{q}_{i+7n+1}=[2i+2n+3,2i-1],\ i=1,...,n-1.$$

The supports are specified by the vectors:

$$\overline{q}_{m-2}=[1,4n+3],\ \overline{q}_{m-1}=[2n+1,4n+4],\ \overline{q}_m=[2n+1,4n+5]. \quad (1)$$

2. Calculating the efforts. Kinematic changeability. A system of equilibrium equations of all the nodes of the girder is written in the matrix form:

$$G\overline{S}=\overline{R}, \quad (2)$$

where $\overline{S}$ is a vector with the length *m* of the efforts in the rods; $\overline{R}$ is a vector of loads.

Vertical external forces applied to the node *i* are written into the even elements of the vector and the horizontal ones into the odd ones. Similarly the even lines of the matrix *G* sized $m\times m$ consist of the direction cosines of efforts with the horizontal axis *x*, the even ones with the vertical axis *y*. In order to calculate a bending, two vectors of loads are necessary:

— $R_{P,2j}=-P$, $j=2,...,2n$, is a load from the forces *P* distributed along the nodes of the lower belt,

— $R_{1,2j}=1$ from a single force applied to the middle node $j=n+1$ of the lower belt (see Fig. 1, 2).

All the remaining elements of the vectors $\overline{R}_P$ and $\overline{R}_1$ are zero.

The solution of the system (2) in the system of symbolic mathematics *Maple* is quicker to get using the inverse matrix method without applying special software packages of the system. The solution is based on the software [5] that is programmed in the *Maple* code. The results are analytical expressions for efforts in all the rods of the girder. In order to identify the formulas for bending that would hold true for a random number of panels, the calculations is first performed sequentially for *n* = 1, 2, 3, … . It is noted that for numbers of panels that is divisible by three, the determinant of the matrix G turns into zero. This indicates kinematic changeability of a structure. In order to ensure that, a scheme of possible speeds at *n* = 3 was found (Fig. 3). In Fig. 3 motionless rods 1—12, 12—8 and 12—4 were distinguished. The rods 2—

11, 11—3 and those symmetrical to them move forward, the rods 2—3 and 10—11 rotate around their centers. The rod 9—10 rotates around a momentary center of velocities, the rods 1—2, 3—4 and 11—12 rotate around one of their ends. The velocities $\vec{u}$ and $\vec{v}$ are connected with the condition $v/a=u/h$ that are due, e.g., an equality of projections of vectors of these velocities onto the rod 3—9.

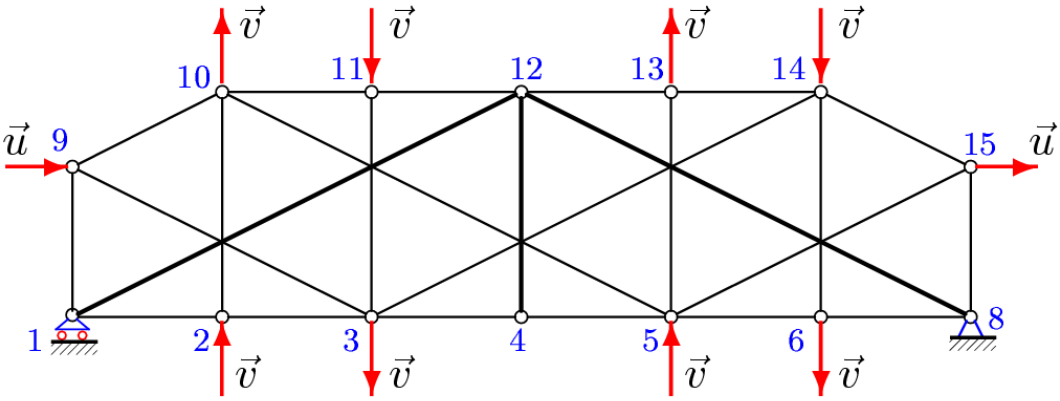

Fig. 3. Scheme of possible speeds

Therefore in order to avoid a number of panels being divisible by three during a sequential solution of problems for girders, a new variable k is introduced that runs through all the natural numbers sequentially so that a corresponding number of panels will only be acceptable:

$$n=(3(2k-1)-(-1)^k)/4,\ \ k=1,2,3,...$$

3. Bending. A vertical displacement of the node where a bending of a girder is evaluated is determined using Moore's integral:

$$\Delta=\sum_{j=1}^{m-3}\frac{S_{P,j}S_{1,j}l_j}{EF}, \tag{3}$$

where $S_{P,j}$ are the efforts in the j-th rod caused by a specified load; $S_{1,j}$ is the effort from a single vertical force applied to the middle angle of the lower belt; E is the elasticity modulus of the rods; F is the area of the section of the rods; l_j is the length of the j-th rod. Summing is conducted along all the rods of the girder expect three rigid supporting ones. The analysis of the resulting bending formula at $k=1,2,...,18$ shows that for any value k the expressions for bending is as follows

$$\Delta EF=P\frac{A_k a^3+B_k h^3+C_k c^3}{2h^2}, \tag{4}$$

where $c=\sqrt{a^2+h^2}$. A difference is only in the coefficients. The dependencies of the coefficients A_k, B_k and C_k on k need to be identified. For that an operator *rgf_findrecur* of the package *genfunc* of the *Maple* system is introduced that allow a recurrent equation for members of a sequence of the coefficients obtained while counting the girders with a different number of panels to be found. E.g., for the coefficient A_k while developing 18 solutions the numbers 1, −2, 24, 45, 217, 352, 910, 1309, … were obtained. The operator yields a corresponding homogeneous ninth-order equation:

$$A_k = A_{k-1}+4A_{k-2}-4A_{k-3}-6A_{k-4}+6A_{k-5}+4A_{k-6}-4A_{k-7}-A_{k-8}+A_{k-9}.$$

For the solution of the equation the operator *rsolve* is used:

$$A_k=(15k^4-10((-1)^k+3)k^3+5(3(-1)^k+1)k^2+2(5-29(-1)^k)k+(53(-1)^k+11)/2)/32.$$

Therefore the sequence of the coefficients at c^3 meets the fifth-order equation:

$$C_k = C_{k-1}+2C_{k-2}-2C_{k-3}-C_{k-4}+C_{k-5}$$

with the solution

$$C_k=3(2k^2-2(1+3(-1)^k)+3(-1)^k+5)/8.$$

The coefficient looks a bit different at h^3:

$$B_k=8k+3(\cos\varphi+\sin\varphi)-4\cos 2\varphi-1,\ \ \varphi=k\pi/2.$$

Checking an analytical solution that takes time to obtain as symbolic transformations are slow already at $k>15$ and are necessary for any k as well as quite large ones, e.g., at $k>100$. Simultaneously comparing the numerical and alalytical results one can prove the effect of accumulating approximation errors in the numerical methods.

Based on the above algorithm, a formula is determined for the displacement of the left moving support of the girder. The displacements are necessary for designing a support structure. The displacement is also calculated using the formula (3), but here $S_{1,j}$ is the effort from a single horizontal force applied to the moving support.

The final formula is as follows

$$\Delta EF = PD_k a^2/h.$$

The coefficient in the formula meets the equation

$$D_k = D_{k-1}+D_{k-2}-D_{k-3}+D_{k-4}-D_{k-5}-D_{k-6}+D_{k-7} \tag{5}$$

with the solution

$$D_k=(6k^2-2(\cos 2\varphi-3)k+\cos 2\varphi-29+12(\cos\varphi+\sin\varphi))/8.$$

3. Analysis. The dependence of bending on the number of panels should be analyzed only if the length of the flight $L = 2na$ is fixed and the total load on the girder is $P_{sum} = (2n-1)P$. Otherwise it is obvious that there will be a growth of bending as the number of panels increases and so does the total load. The suggested way of addressing the problem is in a way similar to a designing task of an optimal choice of the panel length and girder height.

The dependence graph (4) in Fig. 4 is designed for a dimensionless bending $\Delta' = \Delta EF / (PL)$. The coefficients of the resulting dependencies contain "flashing" summands such as $(-1)^k$. In the curved lines there are breaks that are typical of lattice girders [2]. A non-monotonous character of the dependence indicates to an engineer that an optimal choice of the number of panels of a designed structure is possible. As the number of panels increases, the amplitude of leaps goes down and the curved line acquires a constant value. As the height of the girder increases, the bending falls down, which is expected and unpredictable.

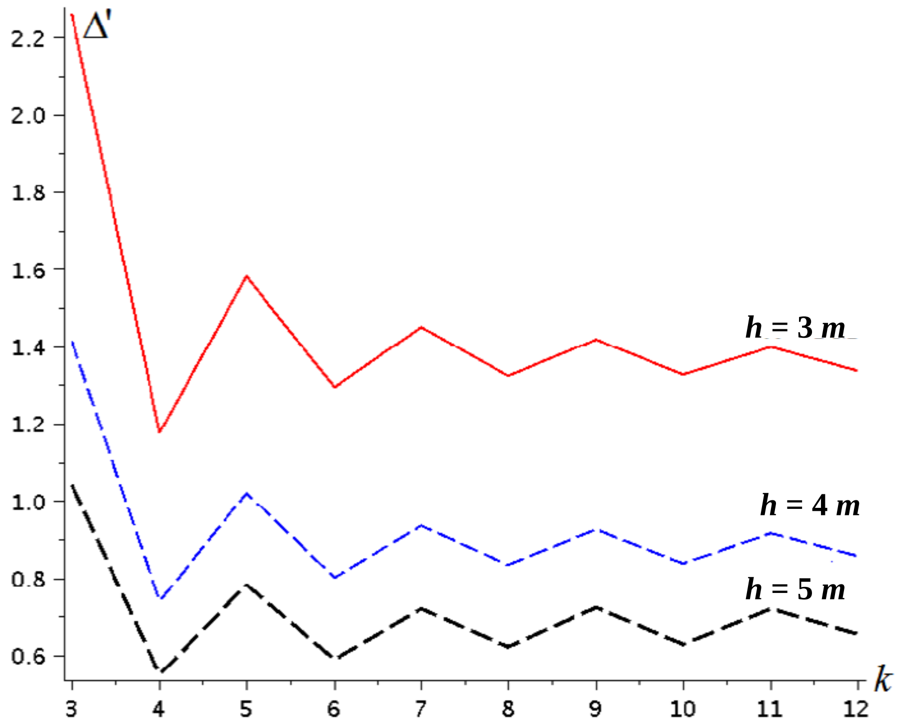

Fig. 4. Bending in the dependence on the number of panels

4. Efforts in the critical rods. An important supplement to a rigidity calculation of a structure is that of the strength of its elements, which means to determine the efforts in the most restrained and stretched rods. In this case it can be done analytically with the formulas of the dependencies of efforts on the number of panels being identified. Obviously under a specified load the most restrained rod whose strength is being checked is in the middle of the upper belt. Based on the above algorithm, the following formula for the efforts in this rod is obtained by means of induction:

$$S^{-} = -P(6ak^2 - 2(3a + a(-1)^k)k + 4a(\cos\varphi + \sin\varphi) - 5a + a(-1)^k)/(16k).$$

The formula for the efforts in the most stretched rod of the middle of the lower belt is as follows:

$$S^{+} = P(6ak^2 - 2(3a + a(-1)^k)k - 4a(\cos\varphi + \sin\varphi) - 5a + a(-1)^k)/(16k).$$

Both formulas with identical modules in their accuracy of up to the sign of one summand are obtained using the solution of the equation (5).

Conclusions

1. The suggested scheme of a statistically determined beam girder accepts a compact analytical solution for bending and efforts in individual rods. The final formulas are easy to use and analyze and contain a sufficient number of varying parameters that characterize a structure and a load. Despite the fact that the solution was obtained for the most common type of load in the engineering practice, the same algorithm can be employed to obtain analytical solutions and other loads. Solutions of similar tasks show that the above load is the most time-consuming for identifying and solving recurrent solutions. Loads as concentrated forces lead to shorter sequences of coefficients based on which the laws of their formation can be found. The analysis of the solution in the graphs showed some features of a structure. The bending turned out to be a non-monotoneous function of a number of panels. There were no asymptotic properties in the solution but visually there is a horizontal line where the solution is for a large number of panels.
2. The resulting formulas can be used as solutions of the tasks of the main system in the method of forces of a statistically undetermined girder option, e.g., that obtained by adding side braces or by introducing joints where rods of the lattice intersect.
3. The described algorithm of identifying the calculation formulas can be employed for more complex tasks as well, e.g., for calculating spatial structures. These solutions already exist [4]. Analytical reviews of the solved tasks of flat girders are in [1, 8]. Evaluations of a bending can be instrumental in a non-linear analysis [15] and optimization of girders [16, 17].

References

1. Kiiko L. K. Analiticheskaya otsenka progiba arochnoi fermy pod deistviem vetrovoi nagruzki [Analytical estimation of arch farm deflection under wind load]. *Nauchnyi vestnik*, 2016, no. 1 (7), pp. 247—254.

2. Kirsanov M. N. Analiz progiba reshetchatoi balochnoi fermy raspornogo tipa [Analysis of the deflection of the lattice girder, the spacer type]. *Inzhenerno-stroitel'nyi zhurnal*, 2015, no. 5 (57), pp. 58— 65.

3. Kirsanov M. N. Staticheskii analiz i montazhnaya skhema ploskoi fermy [Static analysis and flat farm wiring diagram]. *Vestnik Gosudarstvennogo universiteta morskogo i rechnogo flota imeni admirala S. O. Makarova*, 2016, no. 5 (39), pp. 61—68.

4. Kirsanov M. N. Analiticheskoe issledovanie zhestkosti prostranstvennoi staticheski opredelimoi fermy [An analytical study on the stiffness of statically determinate farm]. *Vestnik MGSU*, 2017, vol. 12, iss. 2 (101), pp. 165—171.

5. Kirsanov M. N. *Maple i Maplet. Resheniya zadach mekhaniki* [Maple and Maplet. Solutions of problems of mechanics]. St. Petersburg, Lan' Publ., 2012. 512 p.

6. Kirsanov M. N. *Grafy v Maple* [Graphs in Maple]. Moscow, Fizmatlit Publ., 2007. 168 p.

7. Rybakov L. S. Lineinaya teoriya ploskoi ortogonal'noi reshetki [Linear theory of a planar orthogonal lattice]. Izvestiya RAN. *Mekhanika tverdogo tela*, 1999, no. 4, pp. 174—189.

8. Tin'kov D. V. Sravnitel'nyi analiz analiticheskikh reshenii zadachi o progibe fermennykh konstruktsii [Comparative analysis of analytical solutions to the problem of truss deflection]. *Inzhenerno-stroitel'nyi zhurnal*, 2015, no. 5 (57), pp. 66—73.

9. Al-Shahrabi A. M., Kirsanov M. N. Line of influence of the deflection for cantilever truss / A. M. Al-Shahrabi. *Vestnik nauchnykh konferentsii*, 2016, no. 2—1 (6), pp. 6—7.

10. Dong X., Kirsanov M. N. The dependence of the deflection of the truss from the position of the load for an arbitrary number of panels. *Vestnik nauchnykh konferentsii*, 2016, no. 1—4 (5), pp. 6—7.

11. Galishnikova V. V., Dunaiski P., Pahl P. J. *Geometrically Nonlinear Analysis of Plane Trusses and Frames*. Stellenbosch, SUN MeDIA, 2009. 382 p.

12. Hutchinson R. G., Fleck N. A. Microarchitectured cellular solids — the hunt for statically determinate periodic trusses. *Math. Mech*., 2005, vol. 85, no. 9, pp. 607—617.

13. Hutchinson R. G., Fleck N. A. The structural performance of the periodic truss. *Journal of the Mechanics and Physics of Solids*, 2006, vol. 54, no. 4, pp. 756—782.

14. Jiang H., Kirsanov M. N. An analytical expression for the influence line of the truss. *Vestnik nauchnykh konferentsii*, 2016, no. 1—5 (5), pp. 10—11.

15. Rakhmatulina A. R., Smirnova A. A. The dependence of the deflection of the arched truss loaded on the upper belt, on the number of panels. *Science almanac*, 2017, no. 2—3 (28), pp. 268—271.

16. Richardson J. N., Coelho R. F., Adriaenssens S. Robust topology optimization of truss structures with random loading and material properties: a multiobjective perspective. *Computers & Structures*, 2015, vol. 154, no. 1, pp. 41—47.

17. Tinkov D. V., Safonov A. A. Design Optimization of Truss Bridge Structures of Composite Materials. *Journal of Machinery Manufacture and Reliability*, 2017, vol. 46, no 1, pp. 46—52.

18. Vasilchenko D. I. Analytical evaluation of displacement of the movable support in the truss with reinforced cross bars. *Nauchnyi al'manakh*, 2017, no 4—3 (30), pp. 198—201.

19. Voropai R. A., Kazmiruk I. Yu. Analytical study of the horizontal stiffness of the flat statically determinate arch truss. *Vestnik nauchnykh konferentsii*, 2016, no. 2—1 (6), pp. 10—12.

20. Voropai R. A. Analysis of the deflection of the regular truss with cross type lattice. *Nauchnyi al'manakh*, 2016, no. 4—3 (18), pp. 238—240.