DESIGNING AND CONSTRUCTION OF ROADS, SUBWAYS, AIRFIELDS, BRIDGES AND TRANSPORT TUNNELS

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STRESS-STRAIN OF ELEMENTS OF BRIDGE STRUCTURES WITH A VARYING THICKNESS OF WALLS ALONG THE LENGTH

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Statement of the problem. During erection of bridge passageways spans can be thin-walled structures with one fixed transverse section. With elements of bridge structures being fixed in an uncommon way, during assembly the distribution of stresses might differ significantly from stress fields during the operation of the end object. A preliminary calculation of the stress-strain of construction elements during erection allows safe assembly during bridge construction.
Results. Stress fields in a thin-walled weakly conic structure with a rectangular contour of a transverse section solidly fixed along a skewed edge of one support contour with the other one free. Unlike other popular papers, in this one a varying thickness of the walls of a structure along its length is taken into account. The solutions are presented both numerically and analytically using a tool of special functions. Numerical examples of calculations under a transverse force and a concentrated rolling moment applied to a free end of a structure are presented.
Conclusions. While a considered thin-walled structure is being rolled, there is an edge effect both in the plane of the solid fixation due to the tightness of warping of the contour and at the end section due to the conic shape and change in the thickness. Considering warping of the transverse contour due to a bend has little effect on the stress-strain and can thus be neglected in actual designing.

Keywords: stress-strain, elements of bridge structures, varying thickness.

Introduction. During the erection of different building structures in order to optimize their performance during the assembly and operation, it is necessary to provide even distribution of stresses in construction elements under the impact of applied loads and forces. In some cases it is achieved by varying geometric parameters, construction shape, thickness, fastening, etc.

As for bridge spans, during assembly they are commonly a structure that has no support at the ends and there is a console with a fastened transverse section where stresses can be distributed significantly differently from the way they are in an assembled span. The assembly period up to the final stage may take a few years when construction elements are impacted by a variety of force factors and loads as well as a low bending and rolling rigidity at different stages of the assembly. Thus it is necessary to perform corresponding calculations and develop a set of measures to ensure safety of the assembly of construction elements during the erection of bridges or other building structures.

Development of the theory and practical calculation methods for elements of bridge structures as well as thin-walled systems has been addressed in a lot of studies, e.g., [3, 7, 10—12], etc. The current paper looks at the stress-strain of a skew cone structure supported by a transverse and longitudinal set of a regular structure with a rectangular contour of a transverse section that can be singly or multiply connected. Besides, the walls of a structure along its length are supposed to be of a varying thickness. The transverse set (stringers) can also have a varying area of the transverse section. Fig. 1 shows a calculation model of the above structure.

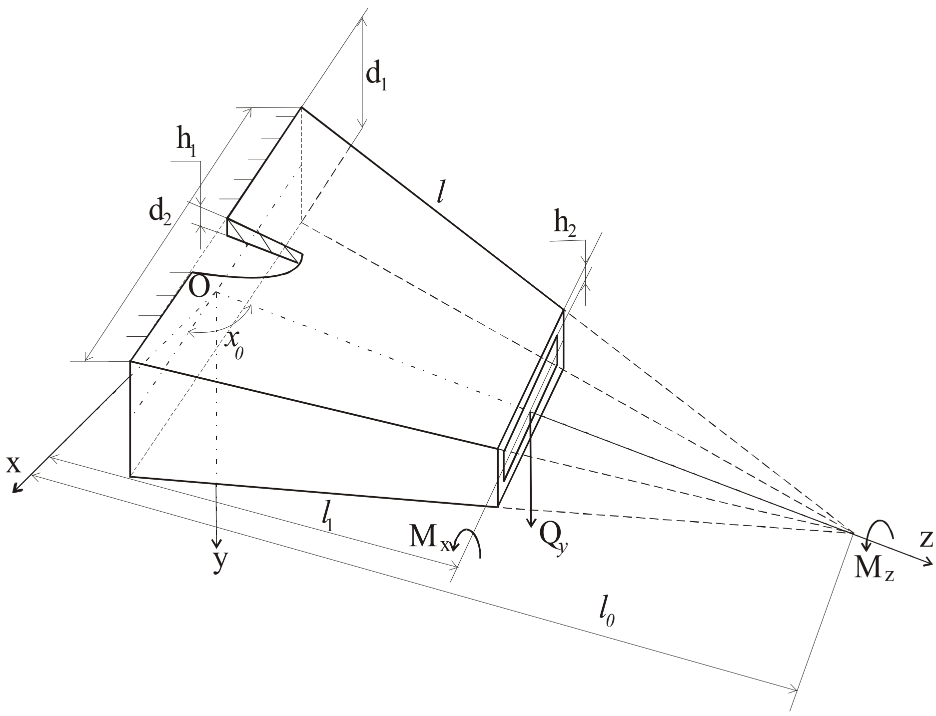

Fig. 1. Calculation model of a construction element

General solutions obtained in the analytical form using a tool of special functions for prismatic structures with a multiply isolated contour of the transverse section are presented in [6, 9], free oscillations are dealt with in [8].

1. Resolving system of regular differential equations and its analytical solution. A system of resolving differential equations

$$\sum_{i=1}^{6+n}\left[\left(\tilde{Z}a_{ji}V_i'+b_{ji}V_i\right)'-b_{ij}V_i'-c_{ji}V_i/\tilde{Z}\right]=R_j/\lambda_jG \qquad (1)$$

and natural boundary conditions

$$\left(\bar{P}_j-P_j\right)\cdot\delta U_j\Big|_{\bar{Z}=0}^{\bar{Z}=Z_1}=0,\ j=1,2,\dots 6+n, \qquad (2)$$

obtained based on the variational principle by Lagrange are assumed to be as in [7]. Here $a_{ji}, b_{ji,} c_{ji}$ are the coefficients that depend on the law of changes in the thickness of the wall of a structure along its length and area of the transverse section of the stringer set; $\tilde{Z}=1-\bar{Z}, \bar{Z}=z/l$ is a relative coordinate along its full length l; R_j is an operator that corresponds with the external load; $V_i=U_i\left(\bar{Z}\right)\lambda_i\left(\bar{Z}\right)$, V_i' are unknown functions and their z derivatives; U_i are initial generalized movements, first three of which are displacements of the contour $\bar{Z}=\text{const}$ as a sold body along the axis Ox, Oy, Oz, U_4, U_5, U_6 are rotation angles of this contour along the axis and the remaining n degress of freedom determine warping of the contour; $\lambda_j=\tilde{Z}$ for $j=4,5,6$; $\lambda_j=1$ for the other values of j; G is a shear modulus; $\bar{P}_j, P_j$ are specified and unknown generalized forces in these sections of a structure; δU_j are variations of the generalized movements.

Let us assume that the pole of the contour $\bar{Z}=\text{const}$ is a point of the intersection of a plane of this contour with the axis *Oz*. Then a vector function $\vec{\phi}_i(S)$ that is responsible for the displacement of the contour $\bar{Z}=\text{const}$ as a solid body in the resolution of an elastic movement vector

$$\vec{U}\left(\bar{Z},S\right)=\sum_{i=1}^{6+n}U_i\left(\bar{Z}\right)\lambda_i\left(\bar{Z}\right)\vec{\phi}_i\left(S\right),$$

is determined using the expressions

$$\vec{\phi}_1=\vec{i},\quad \vec{\phi}_2=\vec{j},\quad \vec{\phi}_3=\vec{k},\quad \vec{\phi}_4=y_0\vec{k}-x_0 ctg\chi_0\vec{j},$$

$$\vec{\phi}_5=x_0(ctg\chi_0\vec{i}-\vec{k}),\quad \vec{\phi}_6=x_0\vec{j}-y_0\vec{i}.$$

The components of the coordinate vector functions $\vec{\phi}_7$ and $\vec{\phi}_8$ associated with warping of the contour $\bar{Z}=\text{const}$ caused by bending and rolling respectively are as follows

$$\phi_{71}=x_0y_0,\quad \phi_{81}=x_0^2y_0+\kappa y_0,$$

$$\phi_{72}=\phi_{7n}=0,\quad \phi_{82}=\phi_{8n}=0,$$

where κ is the coefficient of orthogonalization:

$$\kappa = \oint x_0^2 y_0 dS \Big/ \oint y_0^2 dS.$$

Moving to the absolute coordinate $z = l_0 \bar{Z}$, let us present (1) as

$$\begin{aligned}
&\sum_{i=1}^{6+n}\left(\tilde{Z}a_{ji}V_i' + b_{ji}V_i\right) = \bar{P}_j / \lambda_j G, (j = 1,2,...,6);\\
&\sum_{i=1}^{6+n}\left[\left(\tilde{Z}a_{ji}V_i' + b_{ji}V_i\right)' - b_{ij}V_i' - c_{ji}V_i / \tilde{Z}\right] = -R_j / \lambda_j G, (j = 7,8,...,6+n),
\end{aligned} \tag{3}$$

where $\bar{P}_j$ for j = 1, 2, 3 are the components of the main vector of the external load applied to the cut off part of a structure and for j = 4, 5, 6 are the components of the main moment of external forces in relation to the top of a conic surface. Resolving the system of equations (3) in relation to the initial unknown values and adjusting the solutions to the boundary conditions (2), it is easy to find the constant integrations of the task. E.g., if the contour $\bar{Z} = \text{const}$ has 6 + n degrees of freedom, the boundary conditions are reduced to 6 + n kinematic and static boundary conditions at each face of the shell. If a section is rigidly fastened, kinematic boundary conditions are reduced to the equation to the zero of generalized movements:

$$U_j\Big|_{\bar{Z}=0} = 0, (j = 1,2,...,6+n), \tag{4}$$

i.e. in this case the variation $\delta U_j\Big|_{\bar{Z}=0} = 0$ and (2) for the section $\bar{Z} = 0$ is met. Assuming that in the end section $z = l_1$ there are concentrated force factors, (2) yields the equation $P_j = P_j^*$ that if when expanded, yields

$$G\lambda_j\left(\tilde{Z}\sum_{i=1}^{6+n} a_{ji}V_i' + \sum_{i=1}^{6+n} b_{ji}V_i\right)\Bigg|_{Z=Z_1} = P_j^*(Z_1). \tag{5}$$

Note that the generalized forces P_j are a sum of the operation of internal efforts in the section $\bar{Z} = \text{const}$ of a structure on geometrically possible movements determined with the conditions

$$U_j = \begin{cases} 1, j = i, \\ 0, j \neq i. \end{cases}$$

E.g., let in the end section $z = l_1$ of the above structure a concentrated force Q_y and moments M_x, M_z be applied. Then for a weakly conic structure with a rectangular contour of a transverse section the static boundary conditions (5) in the end section $z = l_1$ are reduced to the following system of equations:

$$\xi_1\left(a_{22}V_2'+a_{24}V_4'+a_{26}V_6'+a_{28}V_8'\right)+b_{24}V_4+b_{27}V_7+b_{28}V_8=Q_y/G,$$
$$\xi_1\left(a_{24}V_2'+a_{44}V_4'+a_{46}V_6'+a_{48}V_8'\right)+b_{47}V_7=\left[M_x+Q_y(l_0-z)\right]/G\xi_1,$$
$$\xi_1\left(a_{26}V_2'+a_{46}V_4'+a_{66}V_6'+a_{68}V_8'\right)+b_{67}V_7+b_{68}V_8=M_z/G\xi_1, \quad (6)$$
$$\xi_1 a_{77}V_7'+b_{77}V_7+b_{78}V_8=0,$$
$$\xi_1\left(a_{28}V_2'+a_{48}V_4'+a_{68}V_6'+a_{88}V_8'\right)+b_{87}V_7+b_{88}V_8=0,$$

where
$$\xi_1=1-l_1/l_0.$$

If a contour of a transverse section is not symmetrical as well as when it is affected by all the force factors Q_x,Q_y,Q_z,M_x,M_y,M_z a complete system of equations (3) is solved, then all the generalized movements U_1,U_2,U_3,U_4,U_5,U_6. are identified. Three more equations will be added to the system (6) that are similar to the first three ones that contain a series of coefficients with the indices $\{i,j\}=\{1,3,5\}$.

Let us study the stress-strain of a structure with a rectangular contour of a transverse section that is rigidly fastened along the skew face (see Fig. 1). The solution of the task in a rather general form makes it necessary to integrate the connected system of equations (3). In order to obtain an analytical approximated solution we will further neglect the influence of warping caused by bending. A resulting error is about 10 % in the embedding plane. Further on, the suggested solution will be discussed in the numerical solution of the end problem. Let us assume that the walls of the above structure have a varying thickness according to the degree law

$$h(\varsigma)=(b-\beta\varsigma)^k, \quad (7)$$

where $b=h_1^{1/k}$; $\beta=\left(h_1^{1/k}-h_2^{1/k}\right)/l_1$; ς is a size coordinate that corresponds with $\bar{Z}$ and forms with the Cartesian axis z the angle $\pi/2-\chi_0$.

Considering the non-zero coefficients $a_{ji},b_{ji,}c_{ji}$ of the resolving system (3) for a structure with a rectangular contour of a longitudinal section, the resolving system (3) is as follows

$$\xi a_{22}U_2'+\xi^2 a_{24}U_4'+\xi^2 a_{26}U_6'+b_{27}U_7=Q_y/G,$$
$$\xi a_{42}U_2'+\xi^2 a_{44}U_4'+\xi^2 a_{46}U_6'+b_{47}U_7=[M_x+Q_y(l_0-\varsigma)]/G\xi, \quad (8)$$
$$\xi a_{62}U_2'+\xi^2 a_{64}U_4'+\xi^2 a_{66}U_6'+b_{67}U_7=M_z/G\xi,$$
$$\xi a_{77}U_7''+(\xi a_{77})'U_7'-c_{77}U_7/\xi=b_{27}U_2'+\xi b_{47}U_4'+\xi b_{67}U_6',$$

where $\xi=1-\varsigma/l_0$.

The first three equations (9) are a system of algebraic equations in relation to U_2',U_4',U_6'. Resolving them in relation to all the variables, we get

$$U_2' = \frac{1}{\xi h}\left[L_1 + \frac{L_2 + L_3\varsigma}{\xi}\right] + \frac{L_4}{\xi}U_7, \quad U_4' = \frac{1}{\xi^2 h}\left[L_5 + \frac{L_6 + L_7\varsigma}{\xi}\right] + \frac{L_8}{\xi^2}U_7,$$
$$U_6' = \frac{1}{\xi^2 h}\left[L_9 + \frac{L_{10} + L_{11}\varsigma}{\xi}\right] + \frac{L_{12}}{\xi^2}U_7, \tag{9}$$

where

$$L_1 = \frac{Q_y}{GA_5}(\bar{a}_{44}A_1 - \bar{a}_{46}A_2), \; L_2 = \frac{[M_z A_2 - (M_x + Q_y l_0)A_1]\bar{a}_{24}}{GA_5}, \; L_3 = \frac{Q_y A_1}{GA_5}\bar{a}_{24},$$

$$L_4 = \frac{A_2(\bar{b}_{27}\bar{a}_{64} - \bar{b}_{64}\bar{a}_{24}) - A_1(\bar{b}_{27}\bar{a}_{44} - \bar{b}_{47}\bar{a}_{24})}{A_5}, \; L_5 = \frac{Q_y / G - L_1\bar{a}_{22} - L_9\bar{a}_{26}}{\bar{a}_{24}},$$

$$L_6 = -\frac{L_2\bar{a}_{22} + L_{10}\bar{a}_{26}}{\bar{a}_{24}}, \; L_7 = -\frac{L_3\bar{a}_{22} + L_{11}\bar{a}_{26}}{\bar{a}_{24}}, \; L_8 = \frac{L_4\bar{a}_{22} - L_{12}\bar{a}_{26}}{\bar{a}_{24}}, \; L_9 = \frac{\bar{a}_{24}Q_y / G - A_3 L_1}{A_2},$$

$$L_{10} = \frac{\bar{a}_{24}(M_x + Q_y l_0)/G + A_3 L_2}{A_2}, \; L_{11} = \frac{\bar{a}_{24}Q_y / G - A_3 L_3}{A_2}, \; L_{12} = \frac{A_3 L_4 + (\bar{a}_{24}\bar{b}_{47} - \bar{a}_{44}\bar{b}_{27})}{A_2},$$

$$A_1 = \bar{a}_{26}\bar{a}_{46} - \bar{a}_{24}\bar{a}_{66}, \; A_2 = \bar{a}_{26}\bar{a}_{44} - \bar{a}_{24}\bar{a}_{46}, \; A_3 = \bar{a}_{22}\bar{a}_{44} - \bar{a}_{24}^2, \; A_4 = \bar{a}_{22}\bar{a}_{46} - \bar{a}_{24}\bar{a}_{26},$$

$$A_5 = A_1 A_3 - A_2 A_4,$$

the coefficients $\bar{a}_{ij}, \bar{b}_{ij}$ do not depend on the transverse coordinate as

$$a_{ij} = \bar{a}_{ij}h(\varsigma), b_{ij} = \bar{b}_{ij}h(\varsigma).$$

Integrating (9), we get the movements of the contour $\bar{Z} = const$ as a solid body:

$$U_2 = L_1\int\frac{d\varsigma}{\xi h(\varsigma)} + L_2\int\frac{d\varsigma}{\xi^2 h(\zeta)} + L_3\int\frac{\varsigma d\varsigma}{\xi^2 h(\varsigma)} + L_4\int\frac{U_7}{\xi}d\varsigma + C_1,$$
$$U_4 = L_5\int\frac{d\varsigma}{\xi^2 h(\varsigma)} + L_6\int\frac{d\varsigma}{\xi^3 h(\varsigma)} + L_7\int\frac{\varsigma d\varsigma}{\xi^3 h(\varsigma)} + L_8\int\frac{U_7}{\xi^2}d\varsigma + C_2, \tag{10}$$
$$U_6 = L_9\int\frac{d\varsigma}{\xi^2 h(\varsigma)} + L_{10}\int\frac{d\varsigma}{\xi^3 h(\varsigma)} + L_{11}\int\frac{\varsigma d\varsigma}{\xi^3 h(\varsigma)} + L_{12}\int\frac{U_7}{\xi^2}d\varsigma + C_3.$$

In order to identify the movements U_2, U_4, U_6 in (10), it is necessary to know U_7 that expresses generalized movements that are caused by warping of the contour. The expression for U_7 will be found using the last equation (8) that will be reduced to a non-homogeneous hypergeometric equation:

$$\xi^2(r\xi - 1)U_7'' + [r(k+1)\xi - 1]\xi U_7' + L_{13}l_0^2(r\xi - 1)U_7 =$$
$$= l_0^2(p\xi - q)^{1-k}[L_{14} + L_{15} + l_0(1-\xi)L_{16}]/q\xi, \tag{11}$$

where $p = \beta l_0$, $q = p - b$, $r = p/q$;

$$L_{13}=\frac{\bar{b}_{27}L_4-\bar{b}_{47}L_8-\bar{b}_{67}L_{12}-\bar{c}_{77}}{\bar{a}_{77}},L_{14}=\frac{\bar{b}_{27}L_1+\bar{b}_{47}L_5+\bar{b}_{67}L_9}{\bar{a}_{77}},$$

$$L_{15}=\frac{\bar{b}_{27}L_2+\bar{b}_{47}L_6+\bar{b}_{67}L_{10}}{\bar{a}_{77}},L_{16}=\frac{\bar{b}_{27}L_8+\bar{b}_{47}L_7+\bar{b}_{67}L_{11}}{\bar{a}_{77}}.$$

While solving the homogeneous equation

$$\xi^2(r\xi-1)U_7''+[r(k+1)\xi-1]\xi U_7'+L_{13}l_0^2(r\xi-1)U_7=0, \qquad (12)$$

that corresponds with (11), let us first determine the parameters α,δ,γ of the hypergeometric function using the ratios [2, 5]:

$$\alpha=a_1+b_1,\ \ \delta=a_1+b_2,\ \ \gamma=1-a_2+a_1,$$

where a_1,a_2,b_1,b_2 are the roots of the algebraic equations

$$a^2+L_{13}l_0^2=0,\quad b^2-kb+rL_{13}l_0^2=0.$$

The solution (12) depends on the parameter γ. If this is a noninteger, the solution of the homogeneous equation (12) is as follows

$$U_7^0=\xi^a\left[C_4F(\alpha,\delta,\gamma,r\xi)+C_5\xi^{1-\gamma}F(\alpha-\gamma+1,\delta-\gamma+1,2-\gamma,r)\right],$$

where
$$F(\alpha,\beta,\gamma,r\xi)=1+\sum_{i=1}^{\infty}\frac{[\alpha]_i[\delta]_i}{i![\gamma]_i}(r\xi)^i,$$

$$[\alpha]_i=\Gamma(\alpha+i)/\Gamma(\alpha)=\alpha(\alpha+1)...(\alpha+i-1).$$

If the parameter $\gamma=1+m$ where $m=0,1,...$ and the parameters α and δ are different from $0,1,\ldots m$, the solution of the equation (12) is as follows

$$U_7^0=\xi^a\left[C_4F(\alpha,\delta,\gamma,r\xi)+C_5\Phi(\alpha,\delta,\gamma,r\xi)\right],$$

where
$$\Phi(\alpha,\delta,\gamma,r\xi)=F(\alpha,\delta,\gamma,r\xi)\ln r\xi-\sum_{i=1}^{a-1}\frac{(i-1)![1-\gamma]_i}{[1-\alpha]_i[1-\gamma]_i}(r\xi)^i+$$
$$+\sum_{i=1}^{\infty}\frac{[\alpha]_i[\delta]_i}{i![\gamma]_i}(r\xi)^i\sum_{n=1}^{i}[(\alpha+n-1)^{-1}+(\delta+n-1)^{-1}-(\gamma+n-1)^{-1}-n^{-1}].$$

After determining two linearly independent solutions Ψ_1 and Ψ_2 of the homogeneous equation (12), the variational method of the Lagrange constant, let us find the general integral of the equation (11):

$$U_7=C_4\Psi_1+C_5\Psi_2+\Psi_2\int\frac{\Psi_1R}{\Psi_1\Psi_2'-\Psi_1'\Psi_2}d\varsigma,-\Psi_1\int\frac{\Psi_2R}{\Psi_1\Psi_2'-\Psi_1'\Psi_2}d\varsigma, \qquad (13)$$

where
$$R=l_0^2(b-\beta\varsigma)^{1-k}(A_{14}+A_{15}+A_{16}\varsigma l_0)/(\beta l_0-b)\xi.$$

The hypergeometric rows that are included in the solution for U_7^0 agree at $0<\zeta<2l_0$. At $\zeta=0$ the rows might fall apart. In order for the rows to agree at $\zeta=0$, it is necessary that the condition $\mathrm{Re}(\gamma-\alpha-\delta)>0$ is met. While checking whether the inequality is met, we get that at $k<1$ the row absolutely agree; at $1<k<2$ they do not agree but not absolutely; at $k>2$ the row disagrees. If we are interested in the solution at $k>1$ at the point $\zeta=0$, using the formula

$$F(\alpha,\delta,\gamma,x)=F(\gamma-\alpha,\gamma-\delta,\gamma,x)(1-x)^{\gamma-\alpha-\delta}$$

the hypergeometric function at $\mathrm{Re}(\gamma-\alpha-\delta)<0$ can be transformed into the hypergeometric function at $\mathrm{Re}(\gamma-\alpha-\delta)>0$. The corresponding formula for the second order hypergeometric function is

$$\Phi(\alpha,\delta,\gamma,x)=(1-x)^{\gamma-\alpha-\delta}\left\{\Phi(\gamma-\alpha,\gamma-\delta,\gamma,x)-\left[\frac{1}{\alpha-\gamma+1}+\ldots+\right.\right.$$
$$\left.\left.+\frac{1}{\alpha-1}+\frac{1}{\delta-\gamma+1}+\ldots+\frac{1}{\delta-1}\right]F(\gamma-\alpha,\gamma-\delta,\gamma,x)\right\}.$$

The analysis of the task shows that at $L_{13}>0$ the parameters of the hypergeometric row can be complex numbers and at $L_{13}<0$ real numbers. For the above structures $L_{13}<0$ always holds so cases of complex values of the parameters in the hypergeometric functions are not considered.

The constant integrations $C_1,C_2,\ldots,C_5$ that are included in the solutions (10) and (13) are determined using the boundary conditions (4), (6).

Let us consider a very important case when the thickness of the walls of a structure changes according to the linear law

$$h(\varsigma)=b-\beta\varsigma.$$

Then the last equation of the system (8) will be presented as

$$\xi^2(r\xi-1)U_7''+\xi[2r\xi-1]U_7'+L_{13}l_0^2(r\xi-1)U_7=\frac{l_0^2[L_{14}+L_{15}+l_0(1-\xi)L_{16}]}{q\xi}.$$

By inserting

$$t=r\xi,\eta=U_7t^{-a}$$

the last equation is transformed as

$$t(1-t)\eta''+[\delta_1-(\alpha_1+\alpha_2+1)]\eta'-\alpha_1\alpha_2\eta=-\frac{l_0^2[L_{14}+L_{15}+l_0(1-t/\beta)L_{16}]}{q\xi},$$

where $\alpha_1 = a_1 - b_1, \alpha_2 = a_1 - b_2, \delta_1 = a_1 - b_2 + 1$; a_1, a_2, b_1, b_2 are the roots of the equations

$$a^2 + A_{13} l_0^2 = 0, b^2 - b + A_{13} l_0^2 = 0.$$

The general integral of the above non-homogeneous differential equation is as follows

$$U_7 = \xi^2 \left\{ C_4 F(\alpha, \delta, \gamma, r\xi) + C_5 \left[r\xi^{1-\gamma} F(\alpha - \gamma + 1, \delta - \gamma + 1, 2 - \gamma, r\xi) \right] \right\} + \Psi_0,$$

where $\alpha = a_1 + b_1$, $\delta = a_1 + b_2$, $\gamma = 1 + a_1 - a_2$; Ψ_0 is a particular solution.

Instead of the integral presentation of the particular solution (14) determined by means of the variational method of the Lagrange constants, in this case Ψ_0 can be written using the generalized hypergeometric function ${}_3F_2$ [2]:

$$\Psi_0 = -\frac{l_0^2 (L_{14} + L_{15} + l_0 L_{16})}{r^2 \xi^3 q (3 - \alpha_1)(3 - \alpha_2)} {}_3F_2(3; 4 - \delta_1; 1; 4 - \alpha_1; 4 - \alpha_2; 1/r\xi) +$$
$$+ \frac{l_0^3 L_{16}}{r^2 \xi^2 q (2 - \alpha_1)(2 - \alpha_2)} {}_3F_2(2; 3 - \delta_1; 1; 3 - \alpha_1; 3 - \alpha_2; 1/r\xi).$$

Hence after determining the generalized movements U_2, U_4, U_6, U_7, using the formulas [7], it is possible to identify the deformations and stresses at a random point of a structure.

2. Numerical solution and analysis of the results. As was already noted, the obtained analytical solution does not consider warping of the contour $\bar{Z} = const$ caused by bending. While studying multiply connected straight and skew cylindrical thin-walled systems [6, 9], it was proved that a contribution of these movements into the stress strain is relatively small and can thus be neglected in construction solutions. As for weakly conic structures with a skew cut, their performance under a load is siginificantly different from that of the structures of a cylindrical shape as a conic shape causes what is called the effect of internal restraint. Therefore let us solve the previous task in a more restrained form, i.e. considering warping caused by bending. The structure is the same and loaded at the end section with a force Q_y and moments M_x, M_z. In the section $z = 0$ is a rigid embedding, the contour is rectangular, the thickness of the wall varies according to the degree law (7). Further on we will maintain not four as we did above but five generalized movements: U_2, U_4, U_6, U_7, U_8.

The solution of the resolving system of regular differential equations with corresponding boundary conditions was obtained using a numerical method of orthogonal matching of the end task for a system of the first-order linear regular differential equations [4]. By preliminarily resolving the first, third and fifth equations of the system in relation to $U_2'', U_4'', U_6'', U_8''$, we reduce the system of five second-order differential equations to that of ten first-order equations where orthogonal matching is used. The stability of the calculation was checked by gradually decreasing the integration step until the difference between the two solutions obtained using different steps is about 10^{-4}.

Some results of the numerical calculations for the model of the skew conic structure with a varying thickness with a rectangular contour of the transverse section are presented graphically. Fig. 2, 3 shows the character of the distribution of normal stresses $\sigma_z = f(z)$ along the ribs of the upper panel ($y = -d_1/2$) under the impact of a rolling moment M_z = 392 N·m at the end section (Fig. 2) and the bending force Q_y = 588 N (Fig. 3).

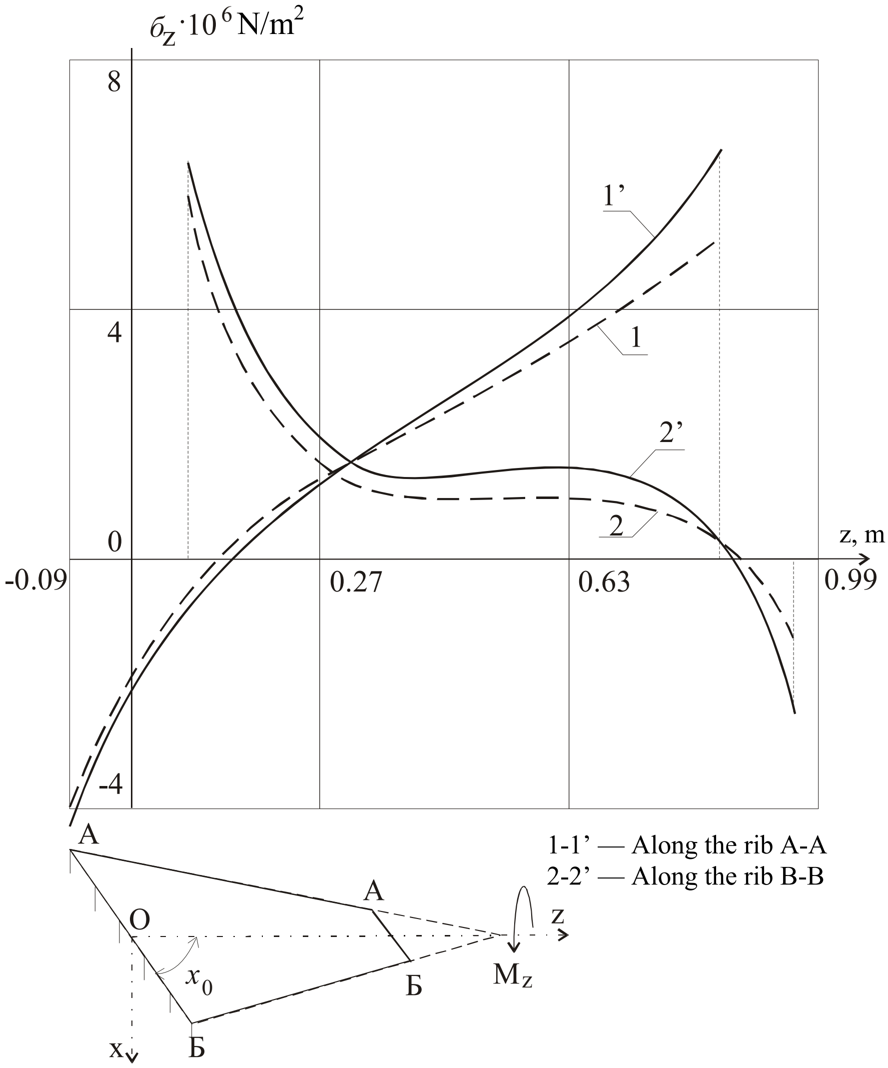

Fig. 2. Distribution of normal stresses under the impact of a rolling moment

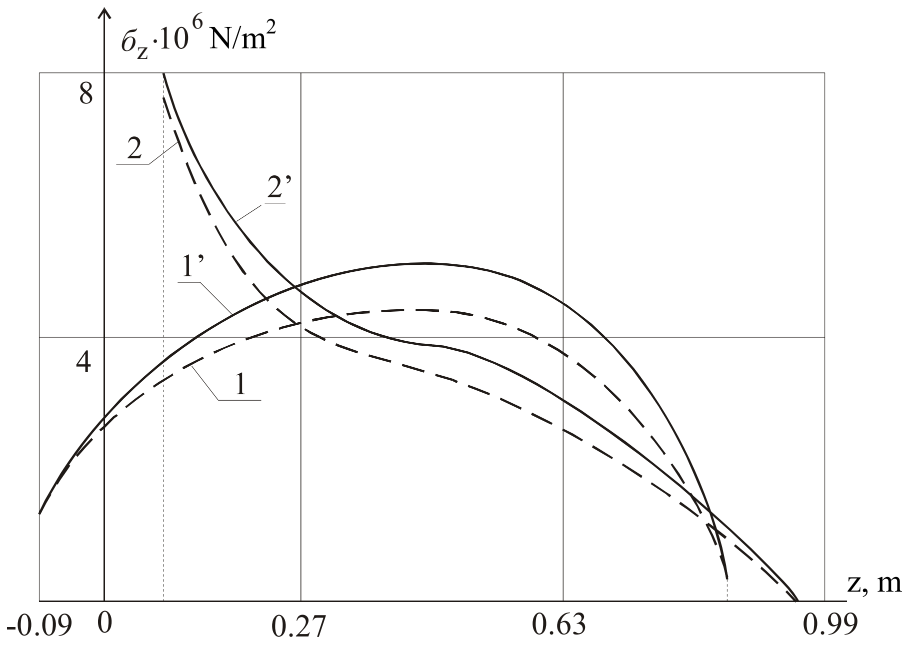

Fig. 3. Distribution of normal stresses under the impact of the bending force

The geometric parameters for the calculations are $l_1 = 0.9$ m, $l_0 = 2$ m, $d_1 = 8 \cdot 10^{-2}$ m, $d_2 = 0.3$ m, $\chi_0 = \pi/3$, $h_1 = 2 \cdot 10^{-3}$ m. The dotted curves are designed for a structure with a constant thickness, the continuous ones are for a structure with a linear law of changes in the thickness at $\overline{h} = h_1 / h_2 = 4/3$. The graphs suggest that during rolling there is an edge effect both in the plane of the rigid embedding due to a restraint in warping of the contour and the end section due to the conic shape and changes in the thickness. As a relative thickness $\overline{h} = h_1 / h_2$ changes, the effect along the ends increases. Obviously, Fig. 3 needs no explanation. As for the effect of warping caused by bending on the stress-strain, as the numerical calculations suggest, it is actually insignificant and can be neglected in actual designing practice. Therefore the graphic dependencies are not presented.

In the limited transition to a structure with a constant thickness, i.e. at $\overline{h} \to 1$, the results of numerical calculations are consistent with those presented in [7] and qualitatitvely agree with the experimental data for a similar structure presented in [1].

3. Solution with the use of the Wentzel-Kramers-Brillouin asymptotic method. Let us go back to the solution of the homogeneous differential equation (12) describing warping caused by rolling. This solution is written using the hypergeometric functions that are endless rows that do not agree much.

Let us now use the Wentzel-Kramers-Brillouin method according to which the solution of the equation (12) will be

$$U_7^0 = \phi(\xi,\gamma)e^{\gamma f(\xi)}, \tag{14}$$

where $\phi(\xi,\gamma)$ is the function of intensity; $f(\xi)$ is the function of changeability.

The function $\phi(\xi,\gamma)$ is approximated using the asymptotic row

$$\phi(\xi,\gamma) \approx \phi_0(\xi) + \phi_1(\xi)/\gamma + ... + \phi_n(\xi)/\gamma^n + ... \; . \tag{15}$$

Inserting (14) considering (15) into the homogeneous equation (12), let us present it as

$$\begin{gathered}\sum_{i=0}^{\infty}\left\{\left[\xi^2\left(\frac{df}{d\xi}\right)^2 - 1\right]\psi\phi_i\gamma^{2-i} + \left\{\left[\xi^2\frac{d^2 f}{d\xi^2} + \xi\left(\frac{\xi rk}{\psi}+1\right)\frac{df}{d\xi}\right]\phi_i + \right.\right.\\ \left.\left. +2\xi^2\frac{df}{d\xi}\frac{d\phi_i}{d\xi}\right\}\psi\gamma^{1-i} + \left[\xi^2\frac{d^2\phi_i}{d\xi^2} + \xi\left(\frac{\xi rk}{\psi}+1\right)\frac{d\phi_i}{d\xi}\right]\psi\gamma^{-i}\right\} = 0,\end{gathered} \tag{16}$$

where $\psi = r\xi - 1$. Making the coefficients for the identical degrees γ in (16) zero, we get an endless system of recurrent equations:

$$\begin{gathered}\left(\frac{df}{d\xi}\right)^2 - \frac{1}{\xi^2} = 0,\\ \left[\xi^2\phi_0\frac{d^2 f}{d\xi^2} + \xi\left(\frac{\xi rk}{\psi}+1\right)\phi_0\frac{df}{d\xi} + 2\xi^2\phi_0'\frac{df}{d\xi}\right]\psi = 0,\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \left\{\left[\xi^2\frac{d^2 f}{d\xi^2} + \xi\left(\frac{\xi rk}{\psi}+1\right)\frac{df}{d\xi}\right]\phi_i + 2\xi^2\frac{df}{d\xi}\frac{d\phi_i}{d\xi} + \xi^2\frac{d^2\phi_{i-1}}{d\xi^2} + \xi\left(\frac{\xi rk}{\psi}+1\right)\frac{d\phi_{i-1}}{d\xi}\right\}\psi = 0.\end{gathered} \tag{17}$$

Solving (17), we will determine the function of changeability $f = f(\xi)$ and intensity function $\phi = \phi(\xi,\gamma)$, i.e. $\phi_0, \phi_1, \phi_2, ..., \phi_i$. It is known that for a fairly large γ in the solution one member of the row will suffice (15), i.e. we can assume that

$$\phi = \phi_0(\xi). \tag{18}$$

Intengrating the first equation of the system (17), we get the expression for the function of changeability:

$$f(\xi) = \pm\ln\xi. \tag{19}$$

Then the second equation of the system (17) considering the solution (19) is significantly more simple:

$$\frac{d\phi_0}{d\xi} = -\frac{rk\phi_0}{2\psi}. \tag{20}$$

Integrating (20), we get the expression of the first member of the row (15):

$$\phi_0 = \psi^{-k/2}. \quad (21)$$

Inserting (19) and (21) into (14), we get a general solution of the homogeneous equation (12):

$$U_7^0 = \psi^{-k/2}(C_4\xi^{\gamma} + C_5\xi^{-\gamma}), \quad (22)$$

where C_4, C_5 are integration constants determined using the boundary conditions of the task. Then using the variational method of the Lagrange constants we get a general integral of a homogeneous hypergeometric equation (12):

$$U_7 = \psi^{-k/2}\left(C_4\xi^{\gamma} + C_5\xi^{-\gamma} + \frac{\xi^{-\gamma}}{2}\int \xi^{1+\gamma}\psi^{k/2}H(\xi)d\xi - \frac{\xi^{\gamma}}{2}\int \xi^{1-\gamma}\psi^{k/2}H(\xi)d\xi \right), \quad (23)$$

where $H(\xi)$ is a function of excitation that is the right part of the equation (11). The obtained solution (23) is much more simple than the above solution using hypergeometric functions and is more convenient to use on PC.

To conclude, let us note that the results of the calculation of the skew structure loaded with a concentrated moment M_z in the section $z = l_1$ obtained using the solution of the hyper-geometric equation and the Wentzel-Kramers-Brillouin method are almost identical. The graphs are as presented in Fig. 2, 3.The latter suggests that the solution using the Wentzel-Kramers-Brillouin method is viable for practical calculations of weakly conic skew systems.

Conclusions. The paper presents the differences between the stress-strain of thin-walled spatial systems with a constant and varying thickness and this should be a consideration in the calculation of actual construction elements of bridge structures.

For the accepted and grounded assumptions the solution of the resolving system of regular differential equations was obtained not only numerically but also analytically using a tool of special functions and the Wentzel-Kramers-Brillouin method. This allows a qualitative analysis of the obtained solutions.

During rolling of the above thin-walled structure there is the edge effect both in the plane of a rigid embedding due to a restraint in warping of the contour and at the end section due to the conic shape and changes in the thickness. Considering warping of the transverse contour caused by bending does not have a significant effect on the stress-strain and can thus be neglected in actual designing practices.

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