# BUILDING STRUCTURES, BUILDINGS AND CONSTRUCTIONS

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## USING THE KANTOROVICH METHOD FOR MODELING OF BENDING VIBRATIONS OF THE FLOATING CONTINUOUS BRIDGE

Bending vibrations of a bridge span are considered within the framework of the general problem on floating continuous bridge vibrations under moving load. The complete solution of the multimode problem by Kantorovich method is given by means of the Rayleigh functions taken into account the effect of vibration damping. The comparative analysis of the Fourier method and Kantorovich method is undertaken.

**Keywords:** bending vibrations of a floating continuous bridge, Kantorovich method, multimode problem, Rayleigh dissipation function, comparative analysis.

## Introduction

In [1] the solution of the problem of vibrations of span of "semi-infinite" pontoon bridge by method of integral transformations and finite-dimensional approach to the description of the dynamics of the pontoon bridge is considered. For the Lagrange function an expression of effective particles with generalized coordinates $q_k(t)$ and masses $m_b$, connected by spring linkages with rigidity $\alpha_{km}$ and influenced by actions of generalized forces $F_k(t)$ has been found as

$$\mathrm{L} = \frac{m_b}{2}\sum_{k=1}^{n}\dot{q}_k^{\,2} - \frac{EJ_z}{2}\sum_{k,m=1}^{n}\alpha_{km}q_k(t)q_m(t) + \sum_{k=1}^{n}F_k(t)q_k(t), \tag{1}$$

where

$$\alpha_{km} = \int_0^l \varphi_k''(x)\varphi_m''(x)dx,\ F_k(t) = \int_0^l F(x,t)\varphi_k(x)dx\cdot$$

Corresponding equations of Euler-Lagrange

$$\frac{d}{dt}\frac{\partial \mathrm{L}}{\partial \dot{q}_i} - \frac{\partial \mathrm{L}}{\partial q_i} = 0 \tag{2}$$

are written as

$$m_b\ddot{q}_i + EJ_z\sum_{k=1}^{n}\alpha_{ik}q_k = F_i(t)\cdot \tag{3}$$

Analyzing the equation (3), we have limited earlier by single-mode approximation. In addition, equations (3) do not take into account the dissipation of energy.

Our goal is improvement of equations (3) purposely to include the effects of vibrations damping, as well as the complete solution of multimode tasks.

## 1. Damping Coefficients

Variation principle is working effectively for conservative forces. To account for the dissipative forces using the Euler-Lagrange equations workaround is required. A so-called dissipative function of Rayleigh is worked in [2, 3]:

$$F = \frac{1}{2}\sum_{i,k}\gamma_{ik}\dot{q}_i\dot{q}_k\ \cdot \tag{4}$$

Corresponding dissipative forces should be added to equations of Euler-Lagrange, so

$$\frac{d}{dt}\frac{\partial \mathrm{L}}{\partial \dot{q}_i} - \frac{\partial \mathrm{L}}{\partial q_i} = -\frac{\partial F}{\partial q_i}. \tag{5}$$

This approach is standard, and dissipative function determine speed of energy dissipation in the system

$$\frac{dE}{dt} = -2F\ . \tag{6}$$

Our problem however originally formulated in the form of partial differential equations:

$$m_б \frac{\partial^2 y(x,t)}{\partial t^2} + \left(1 + k_0 \frac{\partial}{\partial t}\right) EJ_z \frac{\partial^4 y(x,t)}{\partial x^4} = F(x,t) \, . \tag{7}$$

In this equation the rate of energy dissipation is determined by the operator

$$k_0 \frac{\partial}{\partial t} \, .$$

Summand

$$F_{fr} = \left(k_0 \frac{\partial}{\partial t}\right) EJ_z \frac{\partial^4 y(x,t)}{\partial x^4} \tag{8}$$

gives the value of viscous friction forces acting on a unit length of beam.

Since at transition to a finite-dimensional problem expansion into a series is used as

$$y(x,t) = \sum_{k=1}^{n} q_k(t) \varphi_k(x) \, , \tag{9}$$

where

$$\frac{d^4 \varphi_k(x)}{dx^4} = \lambda_k \varphi_k(x) \tag{10}$$

are eigenfunctions of the unrestrained beam of finite length, the specific friction force

$$F_{fr} = \left(k_0 \frac{\partial}{\partial t}\right) EJ_z \frac{\partial^4 y(x,t)}{\partial x^4} = \sum_k k_0 \dot{q}_l EJ_z \lambda_l \varphi_l(x) \, . \tag{11}$$

Power of energy loss through the beam can be obtained from the integration

$$\frac{dE}{dt} = -\int \dot{y}(x,t) F_{fr}(x,t) dx = -\sum_{lm} \int \dot{q}_m(t) \varphi_m(x) k_0 \dot{q}_l EJ_z \lambda_l \varphi_l(x) dx = \\ = -k_0 EJ_z \sum_l \lambda_l \dot{q}_l(t). \tag{12}$$

Obtaining the formula (12) we took into account the orthogonality relation

$$\int \varphi_l(x) \varphi_m(x) dx = \delta_{lm} \, .$$

## 2. Comparative Analysis of the Fourier Method and the Method of Kantorovich

Generally speaking, finite-dimensional system of equations of motion can be obtained without involvement of variation principle, simply substituting expansion for solution in the form of (9) in equation (7), and then equating the coefficients in the orthogonal functions.

This method is the essence of conventional Fourier method. Acting in this way, we obtain the system of equations in the Fourier method:

$$m_b \ddot{q}_l + EJ_z \lambda_l k_0 \dot{q}_l + EJ_z \lambda_l q_l = F_l(t). \tag{13}$$

By virtue of the completeness of systems of functions $\varphi_l(x)$ and convergence of the Fourier series solution in the form of a series (9) converges to the exact solution of equation (7). With all the advantages of relative simplicity of the Fourier method it has the disadvantage — it does not guarantee to obtain the best approximation for the energy of the system while limiting by finite number of term of series in formally exact expansion (9).

If to limit by finite number of term of series and apply the variation principle directly to the Lagrange function in a finite-dimensional representation, the result will differs generally speaking. Taking into account the dissipation the equations of motion in the problem being solved has the form

$$m_b \ddot{q}_l + EJ_z \lambda_l k_0 \dot{q}_l + EJ_z \sum_{m=1}^{n} \alpha_{lm} q_m = F_l(t)\,. \tag{14}$$

If we are interested not in details of behavior of structure, but in more general integral characteristics, this approach (the method of Kantorovich) has unconditional advantage.

It can help us to get the most accurate value of energy

$$H = \frac{m_b}{2} \sum_{k=1}^{n} \dot{q}_k^{\,2} + \frac{EJ_z}{2} \sum_{k,m=1}^{n} \alpha_{km} q_k(t) q_m(t)$$

as a function of time, and on the basis to verify this integral safety test, according to which the strain energy should not exceed the static values of energy corresponding to the parameters for the structures established by sanitary code and regulations for statistically stressed elements.

Both methods, the Fourier method and the method of Kantorovich, give the same results in the case when the functions $\varphi_k(x)$ are at the same time the decisions of the equation (10), containing the fourth derivative, and the equation

$$\frac{d^2\varphi_k(x)}{dx^2} + \sqrt{|\lambda_k|}\varphi_k(x) = 0\,,$$

containing the second derivative of function. It is possible for the harmonic function corresponding to beam supported by both ends, but it doesn’t perform for a unrestrained beam or beam supported by one end. In this case, the methods don’t coincide and give different results at the same choice of expansion basis.

### 3. Solution of Finite-Dimensional System of Equations

The system (14) is connected differential equations. In this case the matrix $\alpha_{lm}$ is real and symmetric. The system of equations (14) as it is well known from the theory of ordinary differential equations [4], could be reduced to a form of system of independent equations by replacing variables $q = Tu$, where matrix $T$ ensures the matrix $\alpha$ reduction to a diagonal form

$$T^{-1}\alpha T = \Lambda, \tag{15}$$

where

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 & 0 & 0 \\ 0 & \Lambda_2 & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \Lambda_n \end{pmatrix}. \tag{16}$$

The transition to normal coordinates $u$ leads to the splittance of the system of equations on independent equations:

$$m_b \ddot{u}_l + EJ_z \lambda_l k_0 \dot{u}_l + EJ_z \Lambda_l u_l = F_l(t). \tag{17}$$

Enter standard notation for further inquiry:

$$\gamma_l = EJ_z \lambda_l k_0 / m_b, \omega_l^2 = EJ_z \Lambda_l / m_b.$$

Give the solution of the equation (17), as detailed, for example, in [5].

The most conveniently the solution is recorded using the Green function $G(t,t')$ satisfying the equation

$$\frac{d^2 G_l}{dt^2} + \gamma_l \frac{dG_l}{dt} + \omega_l^2 G = \delta(t - t'). \tag{18}$$

The solution of equation (18) has the form

$$G_l(t,t') = \theta(t - t') D_l(t - t'), \tag{19}$$

where

$$D_l(\tau) = \frac{1}{\Omega_l} e^{-\gamma\tau/2} \sin(\Omega_l \tau), \quad \theta(\tau) = \begin{cases} 1, \ \tau \geq 0, \\ 0, \ \tau < 0, \end{cases} \tag{20}$$

$\theta(\tau)$ is the discontinuous function of Heavyside,

$$\Omega_l = \sqrt{\omega_l^2 - \left(\frac{\gamma}{2}\right)^2}.$$

Function $D_l(\tau)$ satisfies the homogeneous equation with starting conditions $D_l(0)=0, \dot{D}_l(0)=1$.

Solution of equation (17), corresponding starting conditions, has the form

$$q_l(t_0)=q_{l0}, \dot{q}_l(t_0)=vl_0.$$

In the time point $t_0$ action of external force $F_l(t)$ begins. General solution of equation (17) can be presented in the form

$$q_l(t)=\dot{D}_l(t-t_0)q_{l0}+\gamma D_l(t-t_0)q_{l0}+D_l(t-t_0)v_{l0}+\int_{t_0}^{t}\frac{dt'}{m}G_l(t-t')F_l(t'). \quad (21)$$

The resulting motion is a superposition of vibrations of two types: free and forced.

**Conclusion**

Obtained solutions take into account the damping of oscillations. This leads to two important consequences, actually observed in such systems. First, the restriction of resonant amplitude of oscillations will happen, which otherwise in the exact resonance tends to infinity. Secondly, now the solution at $t>l/v$, that is, after passing the bridge by a pair of wheels, describes damped oscillations, which will cease in the course of time, as it really happens.

Because the real car has a few wheelsets, the effect of the load of each should be taken into account. For one we have already received the solution and discussed its properties. Others wheelsets behave similarly. Therefore, the solution for any other pair will be exactly the same, but with a different value of pressure force on the axis $F_0^j$ and a shift in time of all the solutions on value

$$\Delta t_j = d_j / v,$$

where $d_j$ is the distance from the first wheel pair, for which the moment of entry into the bridge is equal to zero, $v$, as previously, speed of uniform motion of a car. In order to write the kind of solutions for triaxial car, it is necessary to write in the formula (21) an expression for the components of forces sum instead of $F_l(t)$. If the load is a car moving with a constant speed then taking into account several wheelsets

$$F(x,t)=\sum_j F_0^j \delta(x-v(t-\Delta t_j))\sigma(t-\Delta t_j), \quad (22)$$

where $F_0^j$ is the weight of the car, acting on one axis $j$; $x$ is the variable coordinate of the moving car; $\sigma(t)$ is the step function of time, taking into account the beginning of motion of the car on the bridge at the point of time $t = 0$ and the end of movement at the point of time $t = l / v$,

$$\sigma(t) = \begin{cases} 0, & t < 0, \\ 1, & 0 \le t \le l / v, \\ 0, & t > l / v. \end{cases} \tag{23}$$

After integration, taking into account (22) and (23),

$$F_l(t) = \sum_j F_0^{\ j} \varphi_l(v(t - \Delta t_j))\sigma(t - \Delta t_j) / m_b \cdot \tag{24}$$

Then, by virtue of a linear system of equations, it is enough to substitute this expression into equation (14), and then to obtain a solution with the assistance of described above procedure. Similarly, the solution is obtained for a chain of moving cars.

Note that since the functions $\varphi_l$ include harmonic functions, then the force $F_l(t)$ will also include a harmonic summands, resulting in the possibility of excitation of resonance oscillations in the uniform motion of a car on a bridge. Since

$$\lambda_n = \omega_n / a, a^2 = EJ_z / m_b$$

then the frequency of forcing strength $v\lambda_n = \frac{w_n v}{a}$ and natural-vibration frequency of beam is $\omega_{n_1}$.

Thus, condition of the onset of resonance means fulfillment of equality $\frac{v}{a} = \frac{n_1}{n}$. It is desirable to have construction in which fulfillment of this conditionsat commensurable *n, n₁* is unachievable.

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