BUILDING MECHANICS

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DETERMINING THE FIRST FREQUENCY OF THE LONGITUDINAL VIBRATIONS OF A ROD ON A PIECEWISE-HOMOGENEOUS ELASTIC FOUNDATION FORMED BY TWO SECTIONS WITH VARYING STIFFNESS

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Statement of the problem. Despite the introduction into engineering practice of calculation numerical methods into engineering practices, the model «rod on elastic foundation» and «beam on elastic foundation» is still widely used to describe the behavior of structures and facilities. However, most existing research has been performed neglecting the differences in the properties of the foundation. This paper considers the dynamics of the rod of a finite length on a piecewise-homogeneous elastic foundation formed by two sections of different stiffness. The aim of this study is to obtain analytical dependences for the first frequency of the longitudinal vibrations of a rod with different boundary conditions at its ends. Note that the previously conducted similar research for a cantilever rod, at the same time, the task set was solved numerically and only for particular cases without obtaining general analytical dependence, which makes the results difficult to implement in practice.

Results. Using the Ritz method analytical dependences were obtained for the determination of the first frequency of the longitudinal oscillations of a clamped, free and cantilever rod and located on a piecewise-homogeneous elastic foundation. Quantitative evaluation shows good agreement with numerical solutions and those of other researchers.

Conclusions. It is shown that the first natural frequency of longitudinal oscillations is in the trigonometric ratios depending on the lengths of the sections and is in the range specified by the full length of the rod, the stiffness of the foundation and rod, its mass per unit length and the boundary conditions at the ends.

Keywords: rod, elastic foundation, frequency, longitudinal vibrations.

Introduction

Despite the introduction of numerical calculation methods into engineering practices, particularly the finite element method, the models “rod on elastic foundation” and “beam on elastic foun

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dation" is still widely used to describe the behaviour of structures such as railways [2, 3], pipelines [4], extensive foundation [5], etc. Most researchers do not account for the differences in the properties of a sub-base foundation, which is pretty unachievable in practice.

The present paper deals with the dynamics of a finite length rod on a piecewise heterogeneous sub-base foundation formed by two varyingly rigid areas. The aim of the research is to obtain the analytical dependencies to determine the first frequency of eigen longitudinal oscillations of a rod depending on how its ends are fixed.

Note that there have been such studies for a cantilever rod [6] but the task has only been dealt with for particular cases without obtaining a general analytical dependence, which makes the results of the paper challenging to use in practice.

1. Rod on a homogenous rigid foundation. Let us proceed our research with the oscillations of a rod on a homogenous rigid foundation to make it more visual. Let the rod be *L* long and positioned on a foundation with the rigidity *k*.

The equation of longitudinal oscillations of the rod is assumed to be

$$EA\frac{\partial^2 u}{\partial x^2} - m\frac{\partial^2 u}{\partial t^2} - ku = 0, \tag{1}$$

where *EA* is a longitudinal rigidity of the rod; *m* is the linear mass of the rod; *k* is the coefficient of the longitudinal rigidity of an elastic foundation.

The solution (1) will be found using the following [7]

$$u(x,t) = U(x)\left(A\cos\omega t + B\sin\omega t\right), \tag{2}$$

where $U(x)$ is the function of a coordinate x; ω is an original frequency of eigen frequencies; A, B are some constants. After inserting (2) into (1) we write:

$$EA\frac{d^2U}{dx^2} + \left(m\omega^2 - k\right)U(x) = 0. \tag{3}$$

The solution (3) for a high-frequency area for the condition

$$\omega \geq \omega_b = \sqrt{\frac{k}{m}} \tag{4}$$

is as follows

$$U(x) = C\cos\sqrt{\frac{m\omega^2 - k}{EA}}x + D\sin\sqrt{\frac{m\omega^2 - k}{EA}}x, \tag{5}$$

where C and D are some constants; ω_b is a cutoff rate [8].

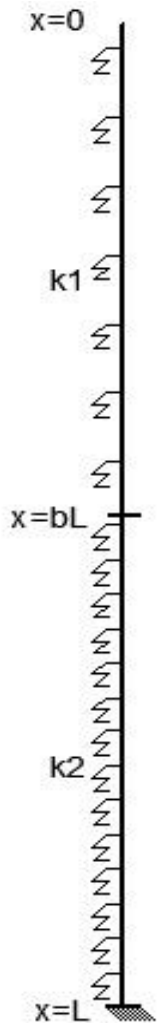


Fig. 1. Statement of the problem

The boundary conditions for a console rod are as follows

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad u(L,t) = 0. \tag{6}$$

Then the expression for the first frequency of eigen oscillations can be the following

$$\Omega = \sqrt{\frac{k}{m} + \frac{\pi^2}{4L^2}\frac{EA}{m}}. \tag{7}$$

Furthermore the boundary conditions for a free rod are as follows

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0, \tag{8}$$

for a restrained one:

$$u(0,t) = 0, \; u(L,t) = 0. \tag{9}$$

Then the expression for the first frequency of eigen oscillations of a free and restrained rod can be as follows

$$\Omega = \sqrt{\frac{k}{m} + \frac{\pi^2}{L^2}\frac{EA}{m}}. \tag{10}$$

2. Console rod. Going back to the subject of the paper, let us proceed with dealing with longitudinal oscillations of the rod where one end is free and the other one is rigidly supported. Let the rod have the length L and be located on a piecewise homogeneous foundation with the rigidity k_1 and k_2 divided by the false coordinate $x = bL$ where $0< b <1$ (Fig. 1).

The equation of longitudinal frequencies of the rod are as shown in (1) and the boundary conditions in (6). Then the expression (5) for each area of the rod is as follows

$$U_1(x) = C_1 \cos\left(\sqrt{\frac{m\omega^2 - k_1}{EA}}x\right), \quad 0 < x < bL, \tag{11}$$

$$U_2(x) = D_2 \sin\left(\sqrt{\frac{m\omega^2 - k_2}{EA}}(L - x)\right), \quad bL < x < L, \tag{12}$$

where C_1 and D_2 are some constants. Taking $x = bL$ as joining points, the conditions of the equality of the movements and deformations of two areas of the rod [9], we obtain the following transcendental equation:

$$tg\left(\sqrt{\frac{m}{EA}}bL\sqrt{\omega^2-\frac{k_1}{m}}\right)tg\left(\sqrt{\frac{m}{EA}}(1-b)L\sqrt{\omega^2-\frac{k_2}{m}}\right)= \\ =\sqrt{\omega^2-\frac{k_2}{m}}\Big/\sqrt{\omega^2-\frac{k_1}{m}}. \tag{13}$$

Note that the solutions (8) are in a range restricted by $b = 0$ and $b = 1$, i.e. when both areas of the rod are in the foundation with k_2 and k_1 respectively. For the task at hand considering (7) for the first frequency of eigen oscillations we write:

$$\Omega_1=\sqrt{\frac{k_1}{m}+\frac{\pi^2}{4L^2}\frac{EA}{m}},\quad b=1, \tag{14}$$

$$\Omega_2=\sqrt{\frac{k_2}{m}+\frac{\pi^2}{4L^2}\frac{EA}{m}},\quad b=0. \tag{15}$$

In order to determine the eigen frequencies of the oscillations ω meeting (13), we use the Ritz method [10, 11].

Taking the expressions for the potential and kinetic energy of the oscillations of the rod on an elastic foundation, according to [9], following the mathematical transformations we get the following system of equations:

$$\frac{\partial}{\partial a_n}\left[\int_0^{bL}\left(\frac{EA}{m}\left(\frac{dU}{dx}\right)^2+\left(\frac{k_1}{m}-\omega^2\right)U(x)^2\right)dx+\int_{bL}^{L}\left(\frac{EA}{m}\left(\frac{dU}{dx}\right)^2+\left(\frac{k_2}{m}-\omega^2\right)U(x)^2\right)dx\right]=0, \tag{16}$$

where a_n are the coefficients of the basic (coordinate) functions $\varphi_n(x)$ meeting the boundary conditions (6). The basic functions are assumed to be eigen longitudinal oscillations of the console rod [11]:

$$\phi_n(x)=\cos\left(\frac{(2n-1)\pi x}{2L}\right),\ \ n=1,2...,N. \tag{17}$$

Similarly to [10, 11] the initial approximation $U(x)$ for determining the eigen frequency of the oscillations we will only use a summand in the Ritz sum:

$$U(x)=a_1\cos\left(\frac{\pi x}{2L}\right). \tag{18}$$

After inserting (18) into (16) and further mathematical transformations we get the following expression for determining the first frequency of eigen oscillations:

$$\omega=\sqrt{\frac{k_2}{m}+\frac{\pi^2}{4L^2}\frac{EA}{m}+\frac{(k_1-k_2)(\sin(\pi b)+\pi b)}{\pi m}}. \tag{19}$$

Now let us proceed to the quantitative evaluation. Let us look at a steel rod with the longitudinal rigidity EA = 4,96e9 Pa, the linear mass m = 185,3 kg/m, length L = 20 m on a piece-

wise homogeneous foundation with k_1 = 10 MPa and k_2 = 20 MPa. The boundary frequencies of the first form of the eigen oscillations are Ω_1 = 522,3 radian/sec and Ω_2 = 467,8 radian/sec. The results of the calculations show that the error in the calculation using (19) is less 0,1 % compared to the numerical solution of the equation (13). There is no need to make any further accuracy assessments.

The dependency $\omega = \omega(b)$ according to (19) is shown in Fig. 2 where the red line corresponds with the conditions of the task at hand at $k_1 < k_2$, the blue one at $k_1 > k_2$ (k_1 = 20 MPa, k_2 = 10 MPa).

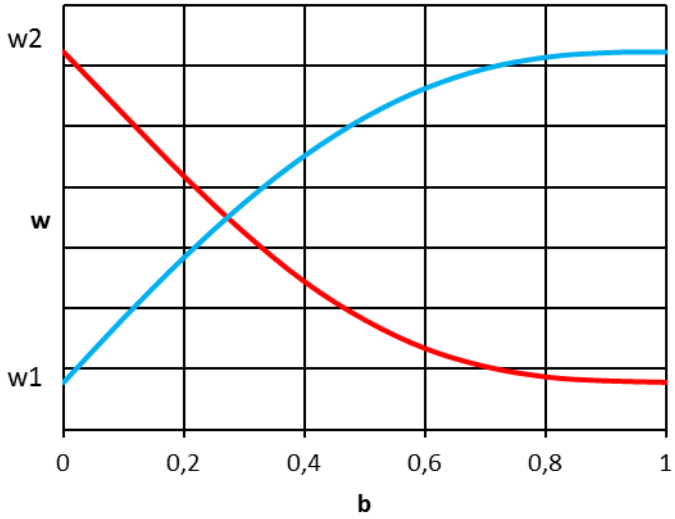


Fig. 2. Graphs of the function $\omega = \omega(b)$

The frequencies computed using the obtained dependence (19) have proved to be highly consistent as compared with the data [6]. Hence the differences for particular cases are no more than 0,01 %.

3. Free rod. Let us move on to the case when both ends of the rod are free (8).

Similarly to the console rod, for the eigen frequencies of the oscillations of the free rod we get the following transcendental equation:

$$-tg\left(\sqrt{\frac{m}{EA}}bL\sqrt{\omega^2-\frac{k_1}{m}}\right)ctg\left(\sqrt{\frac{m}{EA}}(1-b)L\sqrt{\omega^2-\frac{k_2}{m}}\right)=\sqrt{\omega^2-\frac{k_2}{m}}\Big/\sqrt{\omega^2-\frac{k_1}{m}}. \quad (20)$$

The values of the lowest frequency are in the following range of frequency (10):

$$\Omega_1=\sqrt{\frac{k_1}{m}+\frac{\pi^2}{L^2}\frac{EA}{m}}, \quad b=1, \quad (21)$$

$$\Omega_2=\sqrt{\frac{k_2}{m}+\frac{\pi^2}{L^2}\frac{EA}{m}}, \quad b=0. \quad (22)$$

For determining the frequencies we made use of the Ritz method as we previously did [10, 11]. The basic functions $\varphi_n(x)$ meeting (8) are taken to be eigen forms of longitudinal oscillations of a freely supported rod [11]:

$$\phi_n(x) = \cos\left(\frac{n\pi x}{L}\right), \quad n = 1, 2..., N. \tag{23}$$

The initial approximation $U(x)$ for determining the lowest eigen frequencies of oscillations containing the following summand is

$$U(x) = a_1 \cos\left(\frac{\pi x}{L}\right). \tag{24}$$

After inserting (24) into (16) and further mathematical transformations we get the following expressions for determining the first frequency of the eigen frequencies:

$$\omega = \sqrt{\frac{k_2}{m} + \frac{\pi^2}{L^2}\frac{EA}{m} + \frac{(k_1 - k_2)(\sin(2\pi b) + 2\pi b)}{2\pi m}}. \tag{25}$$

The results of the quantitative assessments for the above parameters of the rod show that the error is not more than 0,1 %. compared to the numerical solution (20). There is no need to make any further accuracy assessments. The dependence $\omega = \omega(b)$ according to (23) is shown in Fig. 3 where the red line corresponds with the task at hand at $k_1 < k_2$ and the blue one at $k_1 > k_2$ (k_1 = 20 MPa, k_2 = 10 MPa).

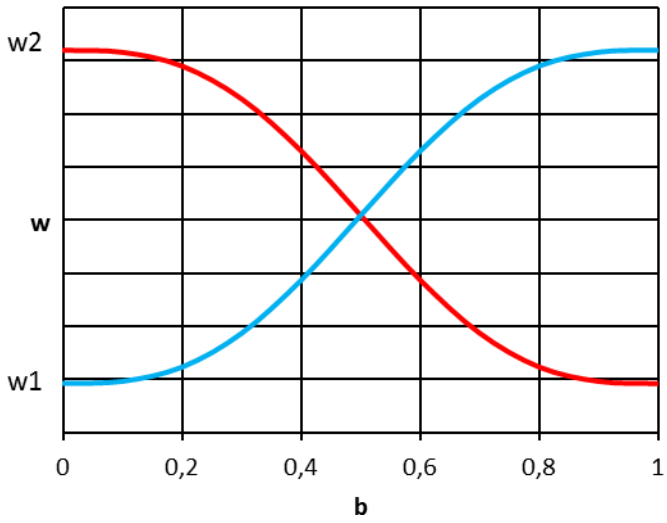


Fig. 3. Graphs of the function $\omega = \omega(b)$

4. Restrained rod. Then let us deal with the conditions for a rigid restriction of both ends of the rod (9). The corresponding frequency equation is as follows

$$-ctg\left(\sqrt{\frac{m}{EA}}bL\sqrt{\omega^2 - \frac{k_1}{m}}\right)tg\left(\sqrt{\frac{m}{EA}}(1-b)L\sqrt{\omega^2 - \frac{k_2}{m}}\right) = \sqrt{\omega^2 - \frac{k_2}{m}} \Big/ \sqrt{\omega^2 - \frac{k_1}{m}}. \tag{26}$$

The lowest frequencies are computed in a range of frequencies restricted by (21) and (22) as they were previously.

The basic functions $\varphi_n(x)$ meeting (9) are taken to be the eigen forms of longitudinal oscillations of a restrained rod [11]:

$$\phi_n(x) = a_n \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2..., N. \tag{27}$$

The initial approximation $U(x)$ for determining the lowest eigen frequencies of oscillations will be the expression containing only one summand:

$$U(x) = a_1 \sin\left(\frac{\pi x}{L}\right). \tag{28}$$

After inserting (28) into (16) and further mathematical transformations we get the following expression for determining the first frequency of eigen oscillations:

$$\omega = \sqrt{\frac{k_2}{m} + \frac{\pi^2}{L^2}\frac{EA}{m} - \frac{(k_1 - k_2)(\sin(2\pi b) - 2\pi b)}{2\pi m}}. \tag{29}$$

The results of the quantitative assessments for the previously investigated parameters of the rod shows that the error is not more than 0,1 % compared to the numerical solution (26) as it previously was. The dependence $\omega = \omega(b)$ according to (29) is shown in Fig. 4 where the red line corresponds with the conditions of the problem at hand at $k_1 < k_2$, the blue one at $k_1 > k_2$ (k_1 = 20 MPa, k_2 = 10 MPa).

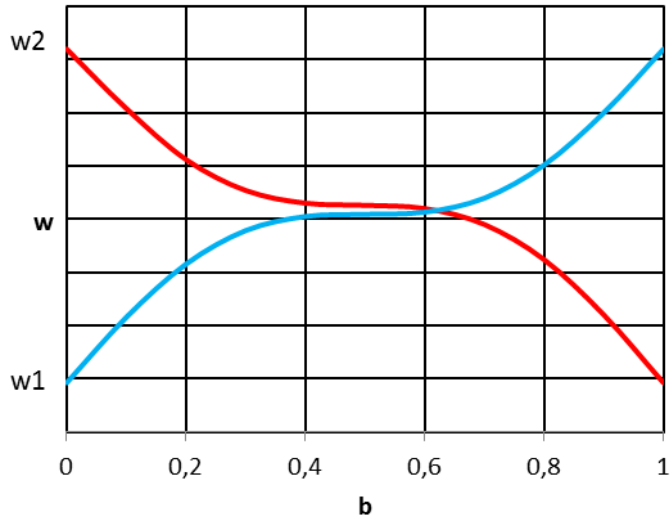


Fig. 4. Graphs of the function $\omega = \omega(b)$

Conclusions

1. The approximated analytical expressions for the first frequencies of the eigen longitudinal oscillations of a restrained, free and console rod on a piecewise homogeneous elastic foundation formed with two areas of varying rigidity. The quantitative assessment shows the results

of the numerical solutions to be in agreement with the data from the previous research [6]. The obtained dependencies can be applied in a particular case as well when a rod is partially restrained into an elastic foundation, i.e. when the rigidity of one of the areas is zero.

2. The first frequency of eigen longitudinal oscillations of a rod on a piecewise homogeneous elastic foundation with two areas of varying rigidity is shown to be non-linearly dependent on the ratios of the lengths of the areas.

3. The first frequency is shown to be in a range specified by the full length of a rod, rigidity of the foudation and that of the rod, its linear mass and boundary conditions at the ends. In order to determine the range of the frequencies simple analytical dependencies were obtained.

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