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DETERMINING THE TEMPERATURE IN A SEMIPLANE WITH A SLANT LINEAR CRACK REACHING THE HALF PLANE BOUNDARY

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Statement of the problem. The paper looks at temperatures in a homogeneous half plane with a finite rectangular crack reaching the half plane boundary on the condition that the temperature at the half plane boundary as well as temperature and heat flow fluctuations in the crack are known.
Results. A mathematical model is suggested that describes a stationary heat distribution in a homogeneous half plane with a linear crack reaching the half plane boundary for when the temperature at the half plane as well as temperature and heat flow fluctuations in the crack are known. The model was proved to be mathematically correct and the method of designing it as well as a whole range of related tasks was presented. The formula for presenting the solution of the model was obtained.
Conclusions. The resulting formula can be employed to study the temperature distribution in a material experiencing cracking as well as in adjacent areas in order to evaluate its effect on heat distribution.

Keywords: temperature, crack, heat flow, heat distribution, equation of stationary heat conductivity.

Introduction. The problem of mathematical description of physical characteristics of materials and defected structures is rather complex and multifaceted (see [6], [7], [12], [17], [18]). This is due to a great variety of materials, structure shapes, types and geometries of defects. One of the aspects of mathematical description of materials and defected structures is to study thermal processes occurring in them (see [3]—[5], [8]—[11], [13]—[16]).

Cracks and other defects cause extra redistribution of heat flows and thus extra strains. Therefore mathematical models describing heat distribution in defected materials and structures can shed some light on the effect of defects on heat flows and temperature distribution.

The paper sets forth a mathematical model for describing heat distribution in a homogeneous half plane with a linear crack to an angle to the half plane boundary. This article is a follow-up of [3], [8], [9], [13] where heat distribution in functional and gradient materials with internal cracking has been investigated. Special attention is paid to proving that the suggested model is mathematically correct as unless it is, the outcome might not relate to the model and only to certain oftentimes rigid input data requirements.

1. Statement of the problem. Major denotations. Let α be a fixed angle in the range of 0 to 180 degrees, $\overline{n}$ is a vector with the coordinates $(-\sin\alpha;\cos\alpha)$.

Let us introduce the denotations: $\mathbb{R}^2_+=\left\{x\in\mathbb{R}^2 \mid x_1\in\mathbb{R},\ x_2>0\right\}$ is the upper half plane, $l_+=\left\{x\in\mathbb{R}^2 \mid x_1=t\cos\alpha,\ x_2=t\sin\alpha,\ \text{where } t\in(0;|l|)\right\}$ is the interval in the upper half plane reaching the half plane boundary, $\overline{l_+}$ is a line corresponding to l_+.

Let us use Δ to denote the Laplace operator in $\mathbb{R}^2$ and $\frac{\partial}{\partial\overline{k}}$ be a derivative in the direction to the vector $\overline{k}$, i.e. if $x=(x_1,x_2)$, $y=(y_1,y_2)$, $\overline{k}=(k_1;k_2)$, then

$$\Delta g(x)=\frac{\partial^2 g(x)}{\partial x_1^2}+\frac{\partial^2 g(x)}{\partial x_2^2},\quad \frac{\partial g(x)}{\partial\overline{k}}=k_1\frac{\partial g(x)}{\partial x_1}+k_2\frac{\partial g(x)}{\partial x_2},\quad \frac{\partial g(x,y)}{\partial\overline{k}_x}=k_1\frac{\partial g(x,y)}{\partial x_1}+k_2\frac{\partial g(x,y)}{\partial x_2}.$$

Let us look at the following task:

$$\Delta u(x)=0,\ x\in\mathbb{R}^2_+\setminus\overline{l_+}, \tag{1.1}$$

$$u(x_1,0)=\psi(x_1),\ x_1\in\mathbb{R}\setminus\{0\}, \tag{1.2}$$

$$u(x+0\cdot\overline{n})-u(x-0\cdot\overline{n})=q_0(x),\ x\in l_+, \tag{1.3}$$

$$\frac{\partial u(x+0\cdot\overline{n})}{\partial\overline{n}}-\frac{\partial u(x-0\cdot\overline{n})}{\partial\overline{n}}=q_1(x),\ x\in l_+. \tag{1.4}$$

The task (1.1)—(1.4) describes a stationary heat distribution in the upper half plane with a cut in the line $\overline{l_+}$ reaching the half plane boundary under the angle α. The line $\overline{l_+}$ models cracking. The equation (1.1) was obtained using the equation of stationary heat distribution in a sold body with no heat sources $div(k(x)grad u(x))=0$ where $k(x)$ is the coefficient of internal heat conductivity, at $k(x)\equiv const\neq 0$. The function $u(x)$ determines the temperature at the point x. The condition (1.2) specifies the temperature at the half plane boundary and the conditions (1.3) and (1.4) specify a temperature and heat flow fluctuation in the crack $\overline{l_+}$ respectively.

The conditions (1.3) and (1.4) are interpreted as follows:

$$u(x+0\cdot\bar{n})-u(x-0\cdot\bar{n})=\lim_{\varepsilon\to+0}(u(x+\varepsilon\cdot\bar{n})-u(x-\varepsilon\cdot\bar{n})),$$

$$\frac{\partial u(x+0\cdot\bar{n})}{\partial\bar{n}}-\frac{\partial u(x-0\cdot\bar{n})}{\partial\bar{n}}=\lim_{\varepsilon\to+0}\left(\frac{\partial u(x+0\cdot\bar{n})}{\partial\bar{n}}-\frac{\partial u(x-0\cdot\bar{n})}{\partial\bar{n}}\right)=$$

$$=\lim_{\varepsilon\to+0}\left[-\sin\alpha\frac{\partial u(x+\varepsilon\cdot\bar{n})}{\partial x_1}+\cos\alpha\frac{\partial u(x+\varepsilon\cdot\bar{n})}{\partial x_2}-\left(-\sin\alpha\frac{\partial u(x-\varepsilon\cdot\bar{n})}{\partial x_1}+\cos\alpha\frac{\partial u(x-\varepsilon\cdot\bar{n})}{\partial x_2}\right)\right].$$

Let A be some set in $\mathbb{R}$ or $\mathbb{R}^2$. $C(A)$ and $C^k(A)$ denote a set of continuous functions and k times continuously differentiated on the set A respectively. $\int_l g(x)dl$ denotes a curved first-order integral of the function $g(x)$ in the curved line l.

We will further assume that the functions $q_0(x)$, $q_1(x)$ are from $C(\overline{l_+})$ and the function $\psi(x_1)$ is from $C(\mathbb{R})$ and restricted in $\mathbb{R}$.

The solution of the task (1.1)—(1.4) will be the function $u(x)$ from $C^2\left(\mathbb{R}^2_+\setminus\overline{l_+}\right)$ which is a classical solution of the equation (1.1) for which the extra conditions (1.2), (1.3) and (1.4) are met.

2. Reducing to the generalized equation. Let $\mathbb{R}^2_-=\left\{x\in\mathbb{R}^2\mid x_1\in\mathbb{R},\ x_2<0\right\}$ denote the lower half plane, $l_-=\left\{x\in\mathbb{R}^2\mid x_1=t\cos\alpha, x_2=-t\sin\alpha, \text{where } t\in(0;|l|)\right\}$ denote the interval in the lower half plane reaching the half plane boundary, $\overline{l_-}$ denotes a line corresponding to l_-.

Let us assume that the task (1.1)—(1.4) has a solution.

Let us look at the function

$$\hat{u}(x)=\begin{cases}u(x_1,x_2), \text{ at } x_2>0,\\ -u(x_1,-x_2), \text{ at } x_2<0,\end{cases} \tag{2.1}$$

where $u(x)=u(x_1,x_2)$ is the solution of the task (1.1)—(1.4).

Immediately using (1.1) and (2.1) we get that in $\left(\mathbb{R}^2_+\setminus\overline{l_+}\right)\cup\left(\mathbb{R}^2_-\setminus\overline{l_-}\right)$

$$\Delta\hat{u}(x)=0. \tag{2.2}$$

Based on (1.3), at $t \in (0;|l|)$.

$$\lim_{\varepsilon\to+0}(u(t\cos\alpha-\varepsilon\sin\alpha,t\sin\alpha+\varepsilon\cos\alpha)-u(t\cos\alpha+\varepsilon\sin\alpha,t\sin\alpha-\varepsilon\cos\alpha))= \\ =q_0(t\cos\alpha,t\sin\alpha). \tag{2.3}$$

Let $\bar{n}_1=(-\sin\alpha;-\cos\alpha)$, $x=(x_1,x_2)\in l_-$, i.e. $x_1=t\cos\alpha$, $x_2=-t\sin\alpha$, where $t\in(0;|l|)$, then considering (2.1) and (2.3)

$$\lim_{\varepsilon\to+0}(\hat{u}(x+0\cdot n_1)-\hat{u}(x-0\cdot n_1))= \\ =-\lim_{\varepsilon\to+0}(u(t\cos\alpha-\varepsilon\sin\alpha,t\sin\alpha+\varepsilon\cos\alpha)-u(t\cos\alpha+\varepsilon\sin\alpha,t\sin\alpha-\varepsilon\cos\alpha))= \\ =-q_0(t\cos\alpha,t\sin\alpha)=-q_0(x_1,-x_2)=\hat{q}_0(x_1,x_2). \tag{2.4}$$

If (1.4) and (2.1) are used, similarly to (2.4) we get that at $x=(x_1,x_2)\in l_-$

$$\frac{\partial\hat{u}(x+0\cdot\bar{n}_1)}{\partial\bar{n}_1}-\frac{\hat{u}(x-0\cdot\bar{n}_1)}{\partial\bar{n}_1}=\hat{q}_1(x), \tag{2.5}$$

where $\hat{q}_1(x)=-q_1(x_1,-x_2)$.

The spaces of infinitely differentiated and finite functions in $\mathbb{R}^2$ and a set of linear and continuous functional over the space will be denoted as $D(\mathbb{R}^2)$ and $D'(\mathbb{R}^2)$ (see [1]).

Let l be a line in $\mathbb{R}^2$, $q(x)$ from $C(\bar{l})$, $\bar{k}=(k_1;k_2)$. $q(x)\delta_l(x)$ and $\dfrac{\partial q(x)\delta_l(x)}{\partial\bar{k}}$ will denote the generalized functions from $D'(\mathbb{R}^2)$, acting according to the following rule: for any function $\varphi(x)$ from $D(\mathbb{R}^2)$

$$\left(q(x)\delta_l(x),\varphi(x)\right)=\int_l q(x)\varphi(x)dl,$$

$$\left(\frac{\partial q(x)\delta_l(x)}{\partial\bar{k}},\varphi(x)\right)=-\int_l q(x)\frac{\partial\varphi(x)}{\partial\bar{k}}dl.$$

Calculating the generalized derivatives from the function $\hat{u}(x)$ in a traditional way considering (2.2), (2.4) and (2.5) we find that the function $\hat{u}(x)$ in the space $D'(\mathbb{R}^2)$ will be the solution of the generalized equation

$$\Delta\hat{u}(x)=2\psi(x_1)\delta'(x_2)+q_1(x)\delta_{l_+}(x)+\frac{\partial q_0(x)\delta_{l_+}(x)}{\partial\bar{n}}+\hat{q}_1(x)\delta_{l_-}(x)+\frac{\partial\hat{q}_0(x)\delta_{l_-}(x)}{\partial\bar{n}_1}, \tag{2.6}$$

where $\bar{n}=(-\sin\alpha;\cos\alpha)$, $\bar{n}_1=(-\sin\alpha;-\cos\alpha)$, $\delta(x_2)$ is the Dirac function (see [1]).

3. Designing the solutions for the generalized equation. In $\mathbb{R}^2$ a fundamental solution of the Laplace operator is the function $\frac{1}{2\pi}\ln|x|$ (see [2]), then the solution of the equation (2.6) is given by the formula

$$\hat{u}(x) = \frac{1}{2\pi}\ln|x| * \\ * \left(2\psi(x_1)\delta'(x_2) + q_1(x)\delta_{l_+}(x) + \frac{\partial q_0(x)\delta_{l_+}(x)}{\partial \overline{n}} + \hat{q}_1(x)\delta_{l_-}(x) + \frac{\partial \hat{q}_0(x)\delta_{l_-}(x)}{\partial \overline{n}_1} \right), \quad (3.1)$$

where $*$ is used for a convolution of the generalized functions (see [1]).
Using the properties of the convolutions of the generalized functions (see[1]), we get that

$$\frac{1}{2\pi}\ln|x| * 2\psi(x_1)\delta'(x_2) = \frac{1}{2\pi}\frac{\partial \ln(x_1^2 + x_2^2)}{\partial x_2} * \psi(x_1)\delta(x_2) = \frac{1}{\pi}\frac{x_2}{x_1^2 + x_2^2} \underset{x_1}{*} \psi(x_1) = \\ = \frac{x_2}{\pi}\int_{-\infty}^{+\infty} \frac{\psi(y_1)}{(x_1 - y_1)^2 + x_2^2}\, dy_1. \quad (3.2)$$

As $\overline{l}$ contains the support of the generalized function $q(x)\delta_l(x)$, for any main function $\varphi(x)$ from $D(\mathbb{R}^2)$

$$(q(x)\delta_l(x) * \frac{1}{2\pi}\ln|x|, \varphi(x)) = (q(x)\delta_l(x) \cdot \frac{1}{2\pi}\ln|y|, \eta(x)\varphi(x+y)),$$

where $\eta(x)$ is a random function from $D(\mathbb{R}^2)$ that is such that $\eta(x) \equiv 1$ in the vicinity of l (see [1]). Determining the direct product of the generalized functions (see [1]) we see that

$$(q(x)\delta_l(x) * \frac{1}{2\pi}\ln|x|, \varphi(x)) = \frac{1}{2\pi}\int_l q(x) \int_{\mathbb{R}^2} \eta(x)\ln|y|\varphi(x+y)dy l_x. \quad (3.3)$$

Using the formula for derivative replacement we get that

$$\int_{\mathbb{R}^2} \ln|y|\varphi(x+y)dy = \int_{\mathbb{R}^2} \ln|z-x|\varphi(z)dz. \quad (3.4)$$

From (3.3) and (3.4) we get that

$$(q(x)\delta_l(x) * \frac{1}{2\pi}\ln|x|, \varphi(x)) = \frac{1}{2\pi}\int_l \int_{\mathbb{R}^2} q(x)\ln|z-x|\varphi(z)dz l_x. \quad (3.5)$$

Changing the order of integration in (3.5) and redenoting z by x and x by y we get

$$(q(x)\delta_l(x) * \frac{1}{2\pi}\ln|x|, \varphi(x)) = \\ = \int_{\mathbb{R}^2} \frac{1}{2\pi}\int_l q(x)\ln|z-x| dl_x \varphi(z)dz = \left(\frac{1}{2\pi}\int_l q(y)\ln|x-y| dl_y, \varphi(x) \right).$$

Thus

$$\frac{1}{2\pi}\ln|x|*q(x)\delta_l(x)=\frac{1}{2\pi}\int_l q(y)\ln|x-y|dl_y. \tag{3.6}$$

Similarly (3.6) we find that

$$\frac{1}{2\pi}\ln|x|*\frac{\partial q(x)\delta_l(x)}{\partial \overline{n}}=\frac{1}{2\pi}\int_l q(y)\frac{\partial \ln|x-y|}{\partial \overline{n}_x}dl_y\ . \tag{3.7}$$

From (3.1), (3.2), (3.6) and (3.7) we get

$$\begin{aligned}\hat{u}(x)=\frac{x_2}{\pi}\int_{-\infty}^{+\infty}\frac{\psi(y_1)}{(y_1-x_1)^2+x_2^2}dy_1+\frac{1}{2\pi}\int_{l_+}q_1(y)\ln|x-y|dl_y+\frac{1}{2\pi}\int_{l_-}\hat{q}_1(y)\ln|x-y|dl_y+\\+\frac{1}{2\pi}\int_{l_+}q_0(y)\frac{\partial\ln|x-y|}{\partial\overline{n}_x}dl_y+\frac{1}{2\pi}\int_{l_-}\hat{q}_0(y)\frac{\partial\ln|x-y|}{\partial\overline{n}_{1x}}dl_y.\end{aligned} \tag{3.8}$$

In (2.4) and (2.5) it was noted that at $x=(x_1,x_2)\in l_-$

$$\hat{q}_0(x)=\hat{q}_0(x_1,x_2)=-q_0(x_1,-x_2);\hat{q}_1(x)=-q_1(x_1,-x_2).$$

Considering the two last equalities (3.8) we get the following

$$\begin{aligned}\hat{u}(x)=\frac{x_2}{\pi}\int_{-\infty}^{\infty}\frac{\psi(y_1)}{(x_1-y_1)^2+x_2^2}dy_1+\frac{1}{2\pi}\int_{l_+}q_1(y)\left(\ln|x-y|-\ln|x-y_-|\right)dl_y+\\+\frac{1}{2\pi}\int_{l_+}q_0(y)\left(\frac{\partial\ln|x-y|}{\partial\overline{n}_x}-\frac{\partial\ln|x-y_-|}{\partial\overline{n}_{1_x}}\right)dl_y,\end{aligned} \tag{3.9}$$

where $\overline{n}=(-\sin\alpha;\cos\alpha),\overline{n_1}=(-\sin\alpha;-\cos\alpha),\ y=(y_1,y_2),\ y_-=(y_1,-y_2)$.

4. Prove of mathematical correctness of the model. Formula for presenting the solution of the model. Note that at this stage we cannot argue that the function $\hat{u}(x)$ is the solution of the task (1.1)—(1.4) as it was obtained assuming that the task (1.1)—(1.4) is correct (that it has the solution). Let us prove that the function $\hat{u}(x)$ given by the equality (3.9) under the previously stated conditions in the function $\psi(x_1)$, $q_0(x)$, $q_1(x)$ will be the solution of the task (1.1)—(1.4), i.e. the solution of the task (1.1)—(1.4) is given by the formula:

$$\begin{aligned}u(x)=\frac{x_2}{\pi}\int_{-\infty}^{\infty}\frac{\psi(y_1)}{(x_1-y_1)^2+x_2^2}dy_1+\frac{1}{2\pi}\int_{l_+}q_1(y)\left(\ln|x-y|-\ln|x-y_-|\right)dl_y+\\+\frac{1}{2\pi}\int_{l_+}q_0(y)\left(\frac{\partial\ln|x-y|}{\partial\overline{n}_x}-\frac{\partial\ln|x-y_-|}{\partial\overline{n}_{1_x}}\right)dl_y,\end{aligned} \tag{4.1}$$

where $\overline{n}=(-\sin\alpha;\cos\alpha),\overline{n_1}=(-\sin\alpha;-\cos\alpha),\ y=(y_1,y_2),\ y_-=(y_1,-y_2)$.

In [4] it was shown that if the function $f(x)$ is continuous in the entire real line probably except a finite number of points where it has the discontinuity of the first kind, for any number of $\delta > 0$ the following ratio holds true:

$$\lim_{\varepsilon\to+0} \frac{\varepsilon}{\pi} \int_{x_1-\delta}^{x_1+\delta} \frac{f(y_1)}{(x_1-y_1)^2+\varepsilon^2} dy_1 = \frac{f(x_1-0)+f(x_1+0)}{2}, \tag{4.2}$$

where $f(x_1-0) = \lim\limits_{\varepsilon\to+0} f(x_1-\varepsilon)$, $f(x_1+0) = \lim\limits_{\varepsilon\to+0} f(x_1+\varepsilon)$.

Let us show that the function $u(x)$ given by the equation (4.1) meets the conditions (1.2).

It is plain to see that

$$u(x_1,0) = \lim_{x_2\to+0} u(x_1,x_2) =$$
$$= \lim_{x_2\to+0} \frac{x_2}{\pi} \int_{-\infty}^{\infty} \frac{\psi(y_1)}{(x_1-y_1)^2+x_2^2} dy_1 + \lim_{x_2\to+0} \frac{1}{2\pi} \int_{l_+} q_1(y)\left(\ln|x-y| - \ln|x-y_-|\right) dl_y + \tag{4.3}$$
$$+ \lim_{x_2\to+0} \frac{1}{2\pi} \int_{l_+} q_0(y) \left(\frac{\partial \ln|x-y|}{\partial \bar{n}_x} - \frac{\partial \ln|x-y_-|}{\partial \bar{n}_{1_x}} \right) dl_y.$$

The restrictions of the function $\psi(x_1)$ and (4.2) mean that

$$\lim_{x_2\to+0} \frac{x_2}{\pi} \int_{-\infty}^{\infty} \frac{\psi(y_1)}{(x_1-y_1)^2+x_2^2} dy_1 = \lim_{x_2\to+0} \frac{x_2}{\pi} \int_{-\infty}^{x_1-\delta} \frac{\psi(y_1)}{(x_1-y_1)^2+x_2^2} dy_1 +$$
$$+ \lim_{x_2\to+0} \frac{x_2}{\pi} \int_{x_1-\delta}^{x_1+\delta} \frac{\psi(y_1)}{(x_1-y_1)^2+x_2^2} dy_1 + \lim_{x_2\to+0} + \frac{x_2}{\pi} \int_{x_1+\delta}^{\infty} \frac{\psi(y_1)}{(x_1-y_1)^2+x_2^2} dy_1 = \tag{4.4}$$
$$= \lim_{x_2\to+0} \frac{x_2}{\pi} \int_{x_1-\delta}^{x_1+\delta} \frac{\psi(y_1)}{(x_1-y_1)^2+x_2^2} dy_1 = \psi(x_1).$$

It is plain to see that

$$\lim_{x_2\to+0} \frac{1}{2\pi} \int_{l_+} q_1(y)\left(\ln|x-y| - \ln|x-y_-|\right) dl_y =$$
$$= \frac{1}{2\pi} \int_{l_+} q_1(y) \lim_{x_2\to+0} \left(\ln|x-y| - \ln|x-y_-|\right) dl_y = \tag{4.5}$$
$$= \frac{1}{2\pi} \int_{l_+} q_1(y)(\ln\sqrt{(x_1-y_1)^2+(-y_2)^2} - \ln\sqrt{(x_1-y_1)^2+y_2^2} dl_y = 0.$$

It is plain to see that

$$\lim_{x_2\to+0} \int_{l_+} q_0(y) \left(\frac{\partial \ln|x-y|}{\partial \bar{n}_x} - \frac{\partial \ln|x-y_-|}{\partial \bar{n}_{1_x}} \right) dl_y =$$
$$= \int_{l_+} q_0(y) \lim_{x_2\to+0} \left(\frac{\partial \ln|x-y|}{\partial \bar{n}_x} - \frac{\partial \ln|x-y_-|}{\partial \bar{n}_{1_x}} \right) dl_y = 0. \tag{4.6}$$

Using (4.3)—(4.6) we find that

$$u(x_1,0)=\lim_{x_2\to+0}u(x_1,x_2)=\psi(x_1),$$

i.e. the condition (1.2) is met.

Similarly using (4.2) we see that for the function $u(x)$ given by the equation (4.1), the conditions (1.3), (1.4) are met.

Immediately inserting the function $u(x)$ into the equation (1.1) we make sure that it is its solution.

Conclusions. The paper looked into essential and urgent research issues related to mathematical description of heat distribution in defected materials and structures as well as determining how defects in materials and structures influence the temperature of materials and structures.

A mathematical model is suggested that allows the temperature distribution to be identified in the half plane with a rectangular crack reaching the half plane boundary knowing the temperature at the half plane boundary and fluctuations of temperature and heat flows in the crack. Using the methods of the theory of generalized functions mathematical correctness of the model was proved and the formula the solution is given by was obtained.

The resulting formula can be used for analyzing the behavior of the temperature in material with a crack including for identifying singularities in the adjacent areas as well as the effect of cracking on heat distribution. Note that the first of the summands in the formula of presenting the solution shows the temperature in the half plane with no cracking if the temperature at the boundary of the half plane is known while the formula itself can be generalized for a random smooth crack.

References

1. Vladimirov V. S. *Uravneniya matematicheskoi fiziki* [Equations of mathematical physics]. Moscow, Nauka Publ., 1976. 527 p.

2. Vladimirov V. S. , Mikhailov V. P., Vasharin A. A. e. a. *Sbornik zadach po uravneniyam matematicheskoi fiziki* [Collection of problems on equations of mathematical physics]. Moscow, Nauka Publ., 1982. 256 p.

3. Glushko A. V., Ryabenko A. S., Loginova E. A., Petrova V. E. Izuchenie statsionarnogo raspredelenie tepla v ploskosti s treshchinoi pri peremen-nom koeffitsiente vnutrennei teploprovodnosti [Study of stationary heat distribution in a plane with a crack at a variable coefficient of internal thermal conductivity]. *Zhurnal vychislitel'noi matematiki i matematicheskoi fiziki*, 2015, vol. 55, no. 4, pp. 695—703.

4. Glushko A. V., Ryabenko A. S., Chernikova A. S. O statsionarnom raspredelenii tepla v dvukh svyaznykh poluploskostyakh s treshchinoi na granitse [On stationary heat distribution in two connected half-planes with a crack at the boundary]. *Vestnik VGU. Seriya. Fizika. Matematika*, 2015, no. 1, pp. 111—134.

5. Ordyan M. G., Petrova V. E. Zadacha teploprovodnosti dlya bimateriala s sistemoi chastichno teplopronitsaemykh treshchin i teplovym istochnikom [The problem of thermal conductivity for bimaterial with a system of partially heat-permeable cracks and a heat source]. *Vestnik Samarskogo gosudarstvennogo univer-siteta (Estestvennonauchnaya seriya)*, 2009, no. 4 (70), pp. 154—170.

6. Panasyuk, V. V., Savruk M. P., Datsyshin A. P. *Raspredelenie napryazhenii okolo treshchiny v plastinakh i obolochkakh* [Distribution of stresses around cracks in plates and shells], Kiev, Nauk. Dumka Publ., 1976. 445 p.

7. Parton V. Z. *Mekhanika razrusheniya* [Fracture mechanics], Moscow, Nauka Publ., 1990. 240 p.

8. Ryabenko A. S. Asimptoticheskie svoistva resheniya zadachi o statsionarnom raspredelenii tepla v odnorodnoi ploskosti s treshchinoi [Asymptotic properties of the solution of the problem of stationary heat distribution in a homogeneous plane with a crack]. *Vestnik VGU. Seriya. Fizika. Matematika*, 2012, no. 1, pp. 187—194.

9. Ryabenko A. S. O statsionarnom raspredelenii tepla v funktsional'no-gradientnykh materialakh s vnutrennei treshchinoi [On stationary heat distribution in functionally graded materials with internal crack]. *Vestnik Voronezhskogo instituta GPS MChS Rossii*, 2014, vol. 2, pp. 40—45.

10. Ryabenko A. S., Chernikova A. S. O edinstvennosti resheniya zadachi, modeliruyushchei raspredelenie tepla v ploskosti s treshchinoi na styke dvukh materialov [On the uniqueness of the solution of the problem modeling the heat distribution in the plane with a crack at the junction of two materials]. *Vestnik VGU. Seriya. Fizika. Matematika*, 2017, no. 4, pp. 124—133.

11. Chiu Tz-Cheng, Tsai Shang-Wu, Chue Ching-Hwei. Heat conduction in a functionally graded medium with an arbitrarily oriented crack. International Journal of Heat and Mass Transfer, 2013, vol. 67, pp. 514—522.

12. Erdogan F. The crack problem for bonded nonhomogeneous materials under antiplane shear loading. J. Appl. Mech, 1985, vol. 52, pp. 823—828.

13. Glushko A. V., Ryabenko A. S., Petrova V. E., Loginova E. A. Heat distribution in a plane with a crack with a variable coefficient of thermal conductivity. Asymptotic Analysis, 2016, vol. 98, no. 4, pp. 285—307.

14. Lee K. Y., Shul C. W. Determination of the thermal stress intensity factors for an interface crack under vertical uniform heat flow. Eng. Fract. Mech, 1991, vol. 40, no. 6, pp. 1067—1074.

15. Lee K. Y., Park S. J. Thermal stress intensity factors for partially insulated interface crack under uniform heat flow. Eng. Fract. Mech, 1995, vol. 50, no. 4, pp. 475—482.

16. Petrova V., Schauder S. Thermal fracture of a functionally graded/homogeneous bimaterial with a system of cracks. Teoretical and Applied Fracture Mechanics, 2011, vol. 55, pp. 148—157.

17. Sladek J., Sladek V., Zhan Ch. An advanced numerical method for computing elastodyanamic fracture parameters in functionally graded materials. Computational Materials Science, 2005, vol. 32, pp. 532—543.

18. Wang X. D., Meguid S. A. The interaction between an interfacial crack and a microcrack under antiplane loading. International Journal of Fracture, 1996, vol. 76, pp. 263—278.